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CLOSED-LOOP SYSTEM ANALYSIS USING

LYAPUNOV STABILITY THEORY

Publication No.______________

Paul Louis Vergez, Ph.D.

The University of Texas at Austin, 1986

Supervising Professor: Jason Speyer

A special class of closed-loop systems composed of a controller and observer in cascade are analyzed. The plant dynamics are assumed to be linear and time-varying but the system parameters are uncertain. The class of observation functions is restricted to those that can be transformed into a linear structure in the state called pseudo-linear measurements where the coefficient may be an explicit function of the original measurements. If along a given path the state vector is observable, then the estimation error of a linear observer structure can be shown to be asymptotically stable. The emphasis is on deriving and analyzing general Lyapunov functions which indicate system stability or a measure of system performance under parameter vari-

ations. The first Lyapunov function is developed by combining the separate controller and observer Lyapunov functions, both of which are quadratic. This combined Lyapunov function is not valid for all linear, time-varying, closed-loop systems. However, the weightings in the controller performance index are scaled such that the combined Lyapunov function is valid for these systems. Further, this Lyapunov function provides a means for developing a more stable system through an overall design selection of the controller and observer parameters.

A second Lyapunov function is derived to account for the system where the controller is a function of the estimated states. This Lyapunov function is valid for linear, time-varying, closed-loop systems.

A third Lyapunov function is derived to directly account for parameter uncertainties in the system model. This Lyapunov function is very useful in identifying system instabilities, given system parameter variations. All three Lyapunov functions are valid for linear, time-varying, and certain classes of nonlinear systems. For linear, time-varying, finite-time problems, the Lyapunov function derived for system parameter variations is used to provide a measure of system performance given these system variations. This Lyapunov function is also used to provide a measure of system performance for the homing missile guidance problem.

A new control law is developed to improve the performance of the pseudomeasurement observer in the guidance loop. The control law is developed from linear quadratic Gaussian theory to minimize the final relative position states and, in addition, improve the psuedomeasurement observer's performance by increasing the observability Grammian matrix. Because of the linear quadratic nature of the problem, a closed-form solution is obtained. The performance gain is measured by the Lyapunov function.

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SECTION I

INTRODUCTION

1.1 Background

The focus of this research is in two areas. The first part is to develop a means of analyzing the performance of closed-loop systems with an observer in the feedback loop, providing state estimates to the control law. Particular emphasis is placed on the homing missile guidance problem. For this class of problems, the observer is nonlinear [109,129]. The second part is to use the information provided from the stability analysis to design a better guidance law.

1.1.1 Stability Analysis

For linear, time-invariant systems, the eigenvalues of the system matrix can be used to determine stability. For continuous-time systems, the eigenvalues of the system matrix must have negative real parts for the system to remain stable. For discrete-time systems, the eigenvalues of the state transition matrix must remain within the unit circle for the discrete system to remain stable.

For linear, time-varying systems, eigenvalue analysis may not provide useful information. It is possible for the closed-loop system to be unstable even if all the eigenvalues have negative real parts for all $t \geq t_0$. It is also possible for the system to be asymptotically stable even if all the eigenvalues of the closed-loop system matrix are constant and some have positive real parts [139].

There have been efforts to apply eigenvalue analysis to certain classes of linear, time-varying systems. Rosenbrock [110] investigated linear, time-varying systems in which the rate of change of the time-varying elements of the system matrix were sufficiently small. He was able to obtain explicit bounds for the time-varying elements where in the system would remain stable. His study was limited to system matricies that were in canonical form.

In a more recent study, Wu [139] has developed a means of determining the necessary and sufficient conditions for the asymptotic stability of linear, time-varying systems. His work involves the concept of mode vectors. Wu defines mode vectors in terms of the extended eigenpairs (the extended eigenvalues and the extended eigenvectors) of the time-varying system matrix.

For nonlinear, closed-loop systems, eigenvalue analysis involves linearizing the system about some nonsingular operating point. This can be useful for systems with small nonlinearities; however, the stability analysis is only valid in an arbitrarily small region about the point of linearization.

A stability analysis concept which has received much attention is Lyapunov's stability theory. The theory can be applied to the class of linear systems and certain classes of nonlinear systems, as well as certain classes of stochastic systems. Given x(t), an n-dimensional vector, and an initial time, t_0, define $x_n(t)$ to be the nominal x(t). $x_n(t)$ is stable in the sense of Lyapunov if to each $\epsilon>0$, there corresponds a region $\delta(\epsilon,t_0)$, such that for any solution, $x_n(t_0)$, whose distance from $x(t_0)$ is $d[x_n(t_0),x(t_0)]<\delta$, then $d[x_n(t),x(t)]<\epsilon$ for all $t\geq t_0$ [38]. This is known as Lyapunov's first method and is applicable only in a small region near the singularity [38]. This will be discussed in more detail in Section II.

A useful approach to determining system stability is the second method of Lyapunov (or the direct method). This method involves the selection of a generalized scalar potential function, called a Lyapunov function. The selected Lyapunov function is tested for certain conditions that denote stability. Lyapunov functions are not unique for any particular system, and can be difficult to obtain for some systems. In addition, the second method of Lyapunov is only a sufficiency test for stability. The significance of this point is a candidate Lyapunov function that doesn't satisfy the stability conditions, does not provide any information concerning system stability. This means that a different candidate function is needed for the Lyapunov analysis. However, if a valid Lyapunov function can be found, this method provides a powerful stability-analysis tool.

The application of Lyapunov's second method to linear feedback control systems and estimation algorithms has received much attention [17,101]. Moore and Anderson [101] analyzed the stability properties, via Lyapunov's second method, for the linear, discrete-time optimal regulator problem. In the same paper, they developed the stability characteristics of a linear estimator. However, the state estimates are not used in any way in the closed-loop control systems. Song and Speyer [119,120] applied Lyapunov's second method to a class of nonlinear estimation algorithms which are of the modifiable type. Modifiable implies that the nonlinearities in either the system dynamics or measurement model can be manipulated into a linear function of the states.

The application of Lyapunov's second method to analyzing the stability of closed-loop control systems containing an observer in the control/feedback loop has received very little attention. In a recent paper by Geering and Basar [49], a Lyapunov function is identified for the standard linear quadratic regulator problem with a linear, full-order state observer. They identified a Lyapunov equation for this system and used the solution in a cost functional of the form

$$J = q^T V q \tag{1.1}$$

where

$$q = [x^T, e^T]^T \tag{1.2}$$

and x is the true state and e is the observer error. This Lyapunov function is used to show that the linear quadratic regulator problem has a superior control gain for every arbitrary choice of the observer gain if and only if the observer is initialized with the true state [49].

The stability analysis of closed-loop systems with observers in the loop is a very important issue since, in most realistic environments, the full true state information will not be available for the control law. The certainty equivalence principle [39] is the basis for combining the separately designed optimal controller and optimal estimator into a cascaded optimal feedback control system. Given this, is it possible to say that the combination of the Lyapunov functions designed for the separate controller and the observer provides a valid Lyapunov function for the cascaded system? If so, this would be an important result, since much is known about the Lyapunov function for the linear regulator problem and for the linear (or modifiable nonlinear) observer problem. Studies by Anderson and Moore [5] and Song and Speyer [119,120] have shown that valid Lyapunov functions exist for the regulator and the observer, separately. If the combined Lyapunov functions are valid for the cascaded system, then the closed-loop system with the observer in the loop is stable when the regulator is stable and the observer is stable. For linear, time-invariant systems, eigenvalue analysis provides the same results. If the cascaded Lyapunov function is valid, one can make the same types of claims for linear, time-varying systems, and certain classes of nonlinear systems.

If the combination of the two separate Lyapunov functions is valid for the closed-loop system, is it the best possible choice for this system in terms of identifying system stability? If not, is there a better

Lyapunov function for the closed-loop system. These issues are the basis for this research.

In addition, if Lyapunov functions can be found for these cascaded systems, can they be put to some practical use in analyzing stability? For instance, can they be used to determine the effects of parameter variations on the stability of a system over a wide range of parameter changes? If not, can a more sensitive Lyapunov function for the cascaded system be determined? This is one issue which is addressed in this research. In this study, the Lyapunov functions are derived for the cascaded controller/filter system; however, the results are based on noiseless measurements so that the filters can be considered as observers. In addition, since the effort concentrates on the homing missile problem (which is a nonlinear, time-varying, finite-time problem), the Lyapunov functions are used to provide a measure of performance of the system.

1.1.2 Stability of Closed-loop Systems Under Parameter Variations

In the process of designing feedback control systems and estimation algorithms, certain assumptions are made. One of the most important assumptions is that the parameters in the model of the dynamics and the measurement device are accurate representations of the true system. If this is not true, the control and estimation algorithm are no longer optimal. In fact, it is possible for the closed-loop system to be unstable depending on how large the errors are in these parameters. Speyer [122] investigated the stability characteristics of linear time-invariant systems with the estimation algorithm in the loop given that parameter variations exist. Given the dynamic models of the control system and the filter algorithm, he restructured the algorithms to emphasize the modelling errors. Then, given the steady state closed-loop system matrix, he determined the range of parameter variations for which the real part of the eigenvalues of the system matrix remain negative; and thus, maintaining system stability. Speyer was able to identify a range of acceptable variations in the system parameters for which the system remained stable. This type of study provides some very useful information for the design of control and estimation algorithms; however, as pointed out in the previous section, the eigenvalue analysis is only useful for linear time invariant problems.

Kalman and Bertram considered the idea of using the Lyapunov function to determine the effects of parameter variations [70]. Their study was based on the idea that without parameter variations, the Lyapunov function is positive definite and its derivative is negative definite. Then, by introducing a parameter variation term in the systems dynamics, it is be possible that the derivative of the Lyapunov function might not be negative definite for certain ranges of values of variations. The same concept was applied by Song and Speyer [119,120] to a nonlinear (modifiable) estimation algorithm.

Another aspect of this research is to investigate the usefulness of Lyapunov stability theory for linear, time-varying and certain classes of nonlinear systems given these variations in parameters. Again, this effort focuses on the identification of a valid Lyapunov function. Once a Lyapunov function exists for the cascaded system, can it be used to determine stability characteristics of the system when the true system parameters deviate from the assumed system parameters? If the Lyapunov function for the cascaded system is not valid when considering parameter variations, can another Lyapunov function be derived which will better measure the stability or performance of the system when parameter variations exist?

There are four main objectives of this dissertation, which relate to Lyapunov theory. The first objective is to determine if the combination of the separate controller and observer Lyapunov functions represent a valid Lyapunov function for the controller/observer cascaded system. The second objective is to derive a Lyapunov function for the linear time-varying, cascaded system for both the continuous-time and discrete-time problems. The third objective is to apply Lyapunov theory to the problem of system stability, given variations in the parameters of the system dynamics and measurement device. The first step is to use the Lyapunov function derived for the cascaded system. A Lyapunov function is then derived where these parameter modelling errors are emphasized in the models. The fourth objective is to apply these Lyapunov functions to specific examples. Two of the examples involve parameter variation analysis for linear, time-invariant problems, where eigenvalue analysis results are available. This is to determine the validity of these Lyapunov functions. A third example is a linear, time-varying guidance problem. As previously stated, the time-varying problem considered is a finite time problem. The Lyapunov functions is used to provide a measure of performance of the system. Although the dynamics, controller, and observer are linear, the fact that the coefficient matrix in the pseudo-linear measurement is a function of the nonlinear measurements of the states makes the whole loop nonlinear.

1.1.3 Homing Missile Problem with Angle Only Measurements

Application of optimal control theory and optimal estimation theory to the homing missile guidance problem have drawn much attention in recent years [131]. The most commonly used control theory is linear quadratic Gaussian (LQG) theory because it is based on a linear system model and provides a closed form solution.

One of the fundamental problems that has limited the practicality of the LQG control law is the difficulty in obtaining the accurate state information required. The LQG guidance law is a function of missile-to-target position, velocity, and target acceleration. Most present day missiles can obtain a measure of the missile's acceleration through on-board accelerometers. In addition, passive seekers are used to provide a measure of line-of-sight angle and rate. It is obvious that the information required by the control law is not directly available and, therefore, must be estimated.

The estimation algorithm used for this effort is a psuedomeasurement observer (PMO). This involves taking the nonlinear angle measurement model and transforming it to a new measurement model which is linear in the states of the system [103]. This algorithm is presented in detail in Section V.

One of the most difficult and critical problems in the observer design is how to model the target acceleration. The target acceleration cannot be measured directly, and directly effects the rest of the states. Therefore, variations in the target acceleration model parameters will be investigated.

1.2 Missile Observer Performance Improvements Through Optimal Feedback Control

Most guidance and estimation schemes are derived separately and are combined through the separation principle. The guidance law which has received considerable attention is the one derived from linear quadratic Gaussian (LQG) theory. It is a useful theory because it provides a linear, closed-form solution; and at the same time, has demonstrated the potential for significant missile guidance improvements [109]. The guidance law is designed under the assumption that the missile-to-target position, velocity, acceleration, and time-to-go are available and known accurately. With the exception of the missile's acceleration, this information is not available on board a homing missile. To provide the LQG guidance law with an estimate of these values, estimation algorithms developed from Kalman filter theory have been extensively investigated [63, 109].

For missile systems with passive (angle-only) seekers on board, the estimation algorithms have not been very successful in accurately estimating the state information [131]; however, the guidance laws have still been successful in producing small terminal miss distances. The guidance law could produce even smaller terminal miss if the state information were more accurately known.

The intent of this part of the research is to design a LQG guidance law that not only tries to minimize the final miss distance (which it does now), but also to improve the performance of the estimation algorithm. Improving the estimators performance is done by incorporating a term in the performance index which attempts to maximize the observability Grammian matrix of the estimation algorithm.

This approach is similiar to the efforts of Hull, Speyer, Tseng, and Larson [63], in which they developed a guidance law using the LQG performance index which included a term that would maximize the information matrix. The guidance law could not be solved in closed form, requiring the use of a numerical optimization program. The results, however, did show that the guidance law could improve the filter algorithm's performance while attempting to hit the target.

The observability term in the performance index for this effort is based on the Lyapunov functions derived for the linear, time-varying problem. The results of this effort differ from Hull, Speyer, Tseng, and Larson in that a closed form solution is attainable.

1.3 Synopsis

In Section II, the two Lyapunov stability methods are presented in detail. Next, the Lyapunov functions for the separate linear time-varying controller and observer (estimation algorithm) are discussed. These two separate Lyapunov functions are then combined to determine if they represent a valid Lyapunov function for the cascaded system. This is done for both the continuous-time case and the discrete-time case. The discrete-time case is addressed since the majority of future guidance and estimation algorithms will exist on digital computers.

In Section III, a Lyapunov function is derived for the cascaded system, assuming the control law is a linear function of the estimated states. This is accomplished by setting the Lyapunov function equal to the average value of the performance index. The Lyapunov function will be validated through the Lyapunov stability conditions stated in Section II. Again, this is done for both the continuous-time and discrete-time problem.

In Section IV, a Lyapunov function is derived for the cascaded system, taking into account parameter variations. The first step is to identify these parameter uncertainties, and to incorporate them into the linear, time-varying system model. The derivation is the same as in Section III, which is done by equating the Lyapunov function to the optimal return function. The validation process is also the same as in Section III.

In Section V, the LQG guidance law is derived to minimize final miss distance and improve the estimation algorithm performance. The estimation algorithm used is the pseudomeasurement observer.

In Section VI, several applications of these Lyapunov functions are studied by numerical analysis and simulations. Given two linear, time-invariant examples, the three Lyapunov functions are used to determine acceptable ranges of parameter uncertainties in order to maintain system stability. Variations in the control matrix are considered. The results are compared to an eigenvalue analysis of the same system with parameter variations to determine the accuracy of the Lyapunov functions to predict the boundaries of stability. The first example is the scalar cascaded system by Speyer [122] and the second example is a multivariable cascaded system of Doyle and Stein [41].

The next step is to demonstrate the effectiveness of the Lyapunov function for a linear, time-varying system. The first example is a linear quadratic Gaussian guidance problem, where the control law is time varying (a function of time-to-go). The Lyapunov functions are used to analyze system performance given errors in time-to-go and errors in the modelled system dynamic matrix. In the second example,a homing missile guidance system is formed using a pseudomeasurement observer, with angle-only measurements, to estimate the states for the control law. The Lyapunov functions are used to analyze system performance due to errors in target acceleration modelling.

Finally, the usefulness of the LQG guidance law derived to minimize miss distance as well as maximize the observability Grammian matrix of the pseudomeasurement observer is demonstrated. The results are compared to a similar effort by Hull, Speyer, Tseng, and Larson [63].

In Section VII, some of the important results are summarized and conclusions are drawn from these results. Finally, suggestion for future research are discussed.

1.4 Summary of Results

The analytic derivations of the Lyapunov functions in this dissertation are based on a cascaded controller/filter system. The numerical results are generated based on the assumption that the measurements were noiseless, and the filter works as a observer.

The Lyapunov function which consists of adding the controller Lyapunov function by Anderson and Moore [6] to the observer Lyapunov function by Song and Speyer [119,120] is not valid for all controller/observer systems. However, the controller performance index is scaled such that the combined Lyapunov functions are valid without affecting the control gain. Further, this Lyapunov function is used as a means of improving the stability of the controller/observer system through an overall design selection of the controller and observer design parameters.

Since the combined Lyapunov function is not valid for all controller/observer systems, a Lyapunov function is derived for this system. The result is a Lyapunov function which consists of the separate controller and observer Lyapunov functions and an additional term which is a coupling of the system states and the observer errors. This Lyapunov function is valid for all controller/observer systems. However, this Lyapunov function is very sensitive to system parameter variations.

A Lyapunov function is derived to directly account for system parameter variations. This Lyapunov function is very accurate in identifying system stability of the linear, time-invariant system under parameter variations when compared to eigenvalue analysis. This Lyapunov function is also useful in providing a measure of system performance for the linear, time-varying, finite-time problem and the homing missile guidance problem.

The control law which is designed for the missile guidance problem to minimize terminal miss as well as improve the performance of an observer in the loop causes the missile to maneuver in such a way as to increase the observability Grammian matrix of the observer and still hit the target. The results are very close to those by Hull, Speyer, Tseng, and Larson [53]. The Lyapunov function from Section III, which is used as the basis for the derivation of this guidance law, shows an improvement in performance over the linear quadratic Gaussian guidance law. The main contribution is that a closed-loop solution of the control law is obtained.

SECTION II

LYAPUNOV FUNCTION FROM SEPARATE CONTROLLER AND FILTER LYAPUNOV FUNCTIONS

2.1 Introduction

Lyapunov functions have been used for the linear quadratic Gaussian control feedback problems, as well as linear observer problems in order to determine their convergence properties. For the controller, it is assumed that all of the system states are available, without inaccuracies. To satisfy the Lyapunov criteria for stability, the Lyapunov function for the controller is selected as a quadratic function of the true states and the controller Riccati matrix [6]. The Lyapunov function for the filter is selected as a quadratic function of the state estimation errors and the inverse of the filter covariance [121]. This Lyapunov function is also valid for the observer problem.

In a more realistic engagement environment, the true states will not be available for the control law. An observer will be needed (for the deterministic case) to provide estimates of the system states to the controller. This results in a cascaded filter and controller in the feedback loop. This section considers the combination of the separate controller and filter Lyapunov functions for the cascaded, closed-loop system.

First, the basic Lyapunov stability methods will be discussed. Next, the Lyapunov functions for the separate controller and filter will be presented, along with how they satisfy the Lyapunov stability theory. These two separate Lyapunov functions will be combined to determine if they represent a valid Lyapunov function for the cascaded system. This is done for both the continuous-time case and the discrete-time case.

2.2 Lyapunov Stability

Lyapunov stability (unlike eigenvalue analysis) provides a tool for analyzing the convergence properties of linear time-varying systems, nonlinear systems, and stochastic systems [95,139]. This dissertation is limited to deterministic systems only. Stability in the sense of Lyapunov can be stated as follows:

Consider the following deterministic system differential equation

$$\dot{x}(t) = f(x(t),t) \quad , \; x(t_o) = x_o \qquad (2.1)$$

where x(t) is the solution to equation (2.1) and is an n-dimensional state vector, t is time, and f is a bounded function over the time interval. Consider a nominal solution, $x_n(t)$, to equation (2.1). The nominal solution is stable in the sense of Lyapunov [18,54,95] if to each $\epsilon > 0$ (no matter how small), and given t_o, there corresponds a $\delta(\epsilon, t_o)$ such that

$$d[x(t_o) - x_n(t_o)] \leq \delta \qquad (2.2)$$

implies that

$$d[x(t) - x_n(t)] \leq \epsilon \qquad (2.3)$$

for all $t \geq t_o$, where $d[\cdot]$ is a distance measure (Figure 2.1). This is known as Lyapunov's first method, and requires an explicit solution to the differential equation (2.1).

A more useful technique which does not require the solution to the differential equation is Lyapunov's second method. This is accomplished through the selection of a generalized scalar potential function, called a Lyapunov function, V(x(t),t). The sufficient conditions for stability in the sense of Lyapunov over the state space are as follows [18,95]:

i) $V(0,t) = 0$

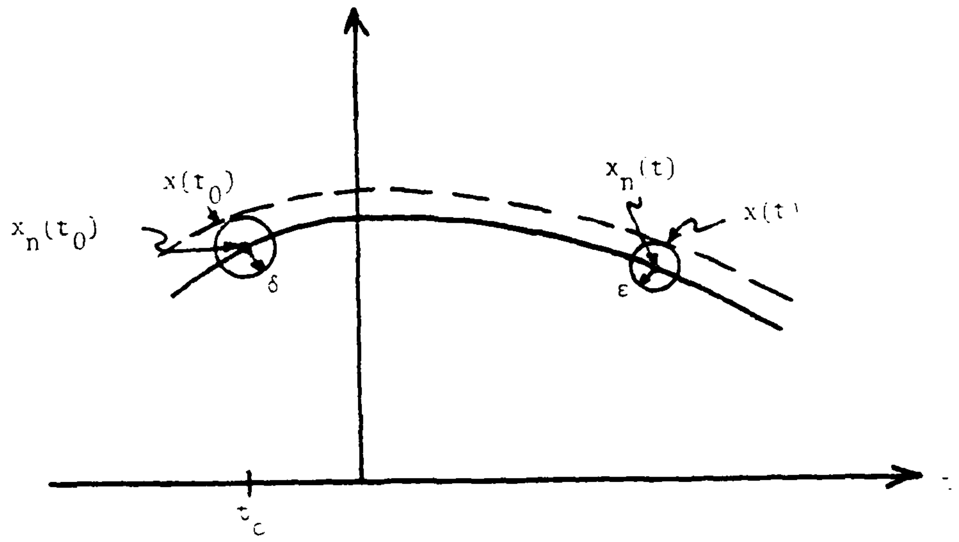


Figure 2.1, Lyapunov Stability

ii) $V(x,t)$ is continuous in both x and t for all $x \in R^n$, and the first partial derivatives in these variables exist.

iii) There exist continuous nondecreasing scalar valued functions, α and β, such that $\beta(||x||) \geq V(x,t) \geq \alpha(||x||) > 0$ for $||x|| \neq 0$.

iv) $\dot{V}(x,t) \leq W(x) \leq 0$ for some continuous, nonpositive $W(\cdot)$, where

$$\dot{V}(x,t) = \frac{\partial V(x,t)}{\partial t} + \sum_{i=1}^{N} \left(f_i(x,t)\frac{\partial V(t,t)}{\partial x_i}\right) \qquad (2.4)$$

where x_i denotes the components of the vector x. If in addition, $W(\cdot)$ is continuous and negative definite or

$\dot{V}(x,t) \neq 0$ except at $x=0$, the solution is asymptotically stable [5,38]. The nominal solution, $x_n(t)$ is said to be asymptotically stable in the sense of Lyapunov if every motion starting sufficiently near $x(t)$ converges to $x(t)$ as $t \to \infty$.

Lyapunov functions are by no means unique, and some functions can provide more meaningful stability results than others [95]. For nonlinear systems, the selection of a useful Lyapunov function is often difficult.

For the case where $f(x(t),t)$ is linear and time-invariant, equation (2.1) becomes

$$\dot{x}(t) = Fx(t) \quad , \quad x(t_o) = x_o \tag{2.5}$$

The stability of this system can be determined by obtaining the eigenvalues of F [38,95]. The system model of equation (2.5) is stable in the sense of Lyapunov if and only if the eigenvalues of F have nonpositive real parts and, to any eigenvalue on the imaginary axis with multiplicity k, there correspond exactly k eigenvectors of F. The system model is asymptotically stable if the eigenvalues have strictly negative real parts.

Another means of determining stability for the time-invariant model is to choose a quadratic Lyapunov function of the form

$$V(x(t)) = x^T(t)Kx(t) \tag{2.6}$$

where K is symmetric and positive definite [38,92,111]. The derivative of V(x(t)) becomes

$$\dot{V}(x(t)) = \dot{x}^T(t)Kx(t) + x^T(t)\dot{K}x(t)$$

$$= x^T(t)\ [F^TK + KF]\ x(t)$$

$$= -\ x^T(t)Qx(t) \tag{2.7}$$

Therefore,

$$Q + F^TK + KF = 0 \tag{2.8}$$

is called the Lyapunov equation.

Choose Q to be positive semidefinite and solve equation (2.8) for K. If K is positive definite then this becomes a necessary and sufficient condition for asymptotic stability of the system in equation (2.5) [93].

For linear, time-varying systems, equation (2.1) becomes

$$\dot{x}(t) = F(t)x(t)\ ,\ x(t_o) = x_o \tag{2.9}$$

Eigenvalue analysis may not provide useful information; however, Lyapunov's second method does apply. Again, choose the Lyapunov function as the following quadratic form

$$V(x(t),t) = x^T(t)K(t)x(t) \quad (2.10)$$

where K(t) is still symmetric and positive definite, and is determined by the following differential Lyapunov equation

$$\dot{K}(t) = -K(t)F(t) - F(t)^T K(t) - Q(t) \;,\; K(t_f) = K_f \quad (2.11)$$

where $Q(t) \geq 0$. For K(t) bounded and positive definite, and $\dot{V}(x(t),t) < 0$ for $x \neq 0$, the system in equation (2.9) is asymptotically stable [96].

If it is desired to analyze the zero-input stability of a nonlinear system model through the linear techniques just described, the nonlinear system model would have to be linearized about some operating point $(x_n(t))$ via a Taylor series expansion, where all nonlinear terms are ignored [95,113]. The linearized version of equation (2.1) is

$$\delta\dot{x}(t) = \left[\frac{\partial f(x,t)}{\partial x} \bigg|_{x=x_n(t)} \right] \delta x(t) \quad (2.12)$$

where $\delta x(t) = x(t) - x_n(t)$ (a perturbation).

The linear stability techniques can be applied to the system model of equation (2.12); however, the analysis is valid only in a small region about the operating point, $x_n(t)$. If the linearized system (2.12) is unstable, the nonlinear system (2.1) is also unstable away from the equilibrium point.

2.3 Linear Quadratic Controller

A linear quadratic controller is designed by minimizing a quadratic performance index for linear systems. Through proper selection of the performance index criteria, useful closed-form solutions of a control law can be derived for linear systems [26]. The quadratic performance index selected for this effort has a terminal constraint on the system, as well as a weighted integral of quadratic terms in the system states and control [26].

2.3.1 Continuous, Time-Varying Lyapunov Function

Given the following linear quadratic optimization problem

$$J = x_F^T G_F x_F$$

$$+ \int_{\emptyset}^{t_f} (x^T(t) Q_c(t) x(t) + u^T(t) R(t) u(t)) dt \quad (2.13)$$

subject to

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \ , \ x(t_o) = x_o \quad (2.14)$$

where x_F is the value of x at the final time. $G_F \geq \emptyset$, $Q_c(t) \geq \emptyset$, $R(t) > \emptyset$, and $A(t)$ and $B(t)$ are linear time-varying matrices. The optimal control, $u(t)$, is a linear function of the states as follows [11]

$$u(t) = - L(t)x(t) \ , \ t \in [t_o, t_f] \quad (2.15)$$

where

$$L(t) = R^{-1}(t) B^T(t) P(t) \quad (2.16)$$

and

$$\dot{P}(t) = -P(t)A(t) - A^T(t)P(t)$$

$$+ L^T(t) R(t) L(t) - Q_c(t) \quad (2.17)$$

$$P(t_F) = G_F \quad (2.18)$$

which is the control Riccati matrix. Equation (2.17) can be rewritten as

$$\dot{P}(t) = - P(t)\bar{A}(t) - \bar{A}^T(t)P(t)$$

$$- L^T(t)R(t)L(t) - Q_c(t) \qquad (2.19)$$

where

$$\bar{A}(t) = A(t) - B(t)L(t) \qquad (2.20)$$

The Lyapunov function that is typically used for this control problem is

$$V(x,t) = x^T(t)P(t)x(t) \qquad (2.21)$$

where P(t) comes from equation (2.19). If (A(t),B(t)) is controllable and $(A(t),Q_c^{1/2}(t))$ is observable [25,84], then P(t) is bounded and positive definite. This satisfies the sufficiency conditions for asymptotic stability in the sense of Lyapunov [121]. Therefore, equation (2.21) represents a good Lyapunov function for the control problem.

Equation (2.21) can be derived by defining the Lyapunov function to be equivalent to the optimal return function, J_o, where

$$V(x,t) = J_o = \min_{u(t)} \{J\} \tag{2.22}$$

2.3.2 Discrete-Time Lyapunov Function

The linear quadratic optimization problem can be solved for the discrete-time problem as follows:

$$J = x_N^T G_N x_N + \sum_{K=\emptyset}^{N-1} x_K^T Q_{c_K} x_K + u_K^T R_K u_K \tag{2.23}$$

subject to

$$x_{K+1} = A_K x_K + B_K u_K \tag{2.24}$$

where $G_N \geq \emptyset$, $Q_{c_K} \geq \emptyset$, $R_K > \emptyset$, and the optimal control is still a linear function of the states and becomes [79]

$$u_K = - L_K x_K \quad ,K = 0,\dots,N \tag{2.25}$$

where

$$L_K = (R_K + B_K^T P_{K+1} B_K)^{-1} B_K^T P_{K+1} A_K \tag{2.26}$$

and

$$P_K = \bar{A}_K^T P_{K+1} \bar{A}_K + L_K^T R_K L_K + Q_{c_K}, \; P_N = G_N \tag{2.27}$$

$$\bar{A}_k = A_k - B_k L_k \qquad (2.28)$$

By equating the discrete Lyapunov function to the discrete optimal return function

$$V_K = J_o = \min_{u_K} \{J\} \qquad (2.29)$$

the Lyapunov function becomes

$$V_k = x_K^T P_K x_K \qquad (2.30)$$

where P_K is symmetric, positive definite by assuming (A_K, B_K) is controllable and $(A_K, Q_{c_K}^{1/2})$ is observable [25,84]. This satisfies the first three conditions for a Lyapunov function. For the discrete-time problem, the fourth condition is replaced by

$$\Delta V_K = V_{K+1} - V_K \leq 0 \qquad (2.31)$$

which becomes

$$\Delta V_K = -x_K^T (L_K^T R_K L_K + Q_{c_K}) x_K \qquad (2.32)$$

With Q_{c_K} positive semidefinite and R_K positive definite, the fourth condition is satisfied; therefore, equation (2.30) is a good Lyapunov function for the linear discrete-time control problem.

2.4 Linear Filter

2.4.1 Continuous, Time-Varying Lyapunov Function

Consider the following model of a linear time-varying filter [48]

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t)$$

$$+ K(t)[y(t) - H(t)\hat{x}(t)] \; , \; \hat{x}(t_0) = \hat{x}_0 \qquad (2.33)$$

where the dynamic system is

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + \Gamma(t)\omega(t) \quad ,$$

$$\omega(t) \sim N(\emptyset, Q_0(t)\delta(t-\tau)) \qquad (2.34)$$

$$y(t) = H(t)x(t) + \mathcal{V}(t) \quad ,$$

$$\mathcal{V}(t) \sim N(\emptyset, R_0(t)\delta(t-\tau)) \qquad (2.35)$$

$$K(t) = \tilde{P}(t)H^T(t)R_0^{-1}(t) \qquad (2.36)$$

$$\dot{\tilde{P}} = A(t)\tilde{P}(t) + \tilde{P}(t)A^T(t)$$

$$- K(t)R_0(t)K^T(t) + Q_0(t) \qquad (2.37)$$

$$\tilde{P}(t_0) = \tilde{P}_\emptyset \qquad (2.38)$$

where $\hat{x}(t)$ is the state estimate, $y(t)$ is the measurement, $A(t)$ is the state dynamics matrix, $H(t)$ is the measurement matrix, $K(t)$ is the Kalman gain, $\tilde{P}(t)$ is the error covariance, and $\nu(t)$ and $\omega(t)$ are Gaussian white noise models where $R_o(t)$ and $Q_o(t)$ are the measurement and state power spectral densities, respectively [48]. The estimation error is given by the equation

$$e(t) = x(t) - \hat{x}(t) \tag{2.39}$$

Differentiating this equation, using equation (2.33) and (2.34), provides the following linear dynamic equation for the estimation error

$$\dot{e}(t) = (A(t) - K(t)H(t))e(t) - K(t)\nu(t) + \Gamma(t)\omega(t) \quad ,$$

$$e(t_\emptyset) = e_\emptyset \tag{2.40}$$

Define the cost functional as

$$J = \int_\emptyset^{t_f} e^T(t)Q_e(t)e(t)dt \tag{2.41}$$

subject to equation (2.40), where $Q_e(t)$ is the weighting on the state error and will be defined later. By using equation (2.22), the Lyapunov function for the linear, time-varying filter problem becomes

$$V(e,t) = e^T(t)\bar{P}(t)e(t) \tag{2.42}$$

where

$$\dot{\bar{P}}(t) = -\bar{P}(t)\tilde{A}(t) - \tilde{A}^T(t)\bar{P}(t) - Q_e(t) \tag{2.43}$$

$$\bar{P}(t_f) = \bar{P}_f \tag{2.44}$$

where

$$\tilde{A}(t) = A(t) - K(t)H(t) \tag{2.45}$$

If $Q_e(t)$ is chosen as

$$Q_e(t) = \tilde{P}^{-1}(t)Q_o(t)\tilde{P}^{-1}(t) + H^T(t)R_o^{-1}(t)H(t) \tag{2.46}$$

then the following identity can be made

$$\bar{P}(t) \equiv \tilde{P}^{-1}(t) \tag{2.47}$$

which can be shown by inverting equation (2.37). Therefore, the Lyapunov function for the observer becomes

$$V(e,t) = e^T(t)\tilde{P}^{-1}(t)e(t) \tag{2.48}$$

which is what is most commonly used. Assuming $(A(t),H(t))$ is observable and $(A(t),\Gamma(t))$ is controllable [25,84], then $\bar{P}(t)$ is bounded and positive definite and equation (2.48) satisfies all four requirements of Section 2.2 to be a valid Lyapunov function for the linear, time-varying filter described in equations (2.33)-(2.38).

The filter algorithm (equations 2.33 to 2.38) can be converted to an observer by changing equation (2.35) to

$$y(t) = H(t)x(t) \tag{2.49}$$

The Lyapunov function (equation 2.48) remains the same for the observer problem.

2.4.2 Discrete-Time Lyapunov Function

The algorithms for the linear, discrete-time filter are

$$\hat{x}_{K+1} = A_K\hat{x}_K + B_K u_K + K_{K+1}[y_K - H_{K+1}\bar{x}_{K+1}] \tag{2.50}$$

$$x_{K+1} = A_K x_K + B_K u_K + \Gamma_K w_K \; , \; w_K \sim N(0, Q_{o_K}) \tag{2.51}$$

$$y_K = H_K x_K + v_K \; , \; v_K \sim N(0, R_{o_K}) \tag{2.52}$$

$$\bar{x}_K = A_{K-1}\hat{x}_{K-1} + B_K u_K \tag{2.53}$$

$$K_K = \tilde{P}_K H_K^T [H_K \tilde{P}_K H_K^T + R_K]^{-1} \tag{2.54}$$

$$\tilde{P}_K = (I-K_KH_K)\bar{P}_K(I-K_KH_K)^T + K_KR_{O_K}K_K^T \tag{2.55}$$

$$\bar{P}_K = A_{K-1}\tilde{P}_{K-1}A_{K-1}^T + Q_{O_K} \tag{2.56}$$

where x_K is the true state, $\hat{x}_K$ is the state estimate, y_K is the measurement, A_K is the state dynamics matrix, B_K is the controller matrix, u_K is the control law, K_K is the Kalman gain, $\tilde{P}_K$ is the covariance matrix, and V_K and ω_K are Gaussian white noise models where R_{O_K} and Q_{O_K} are measurement and state power spectral densities, respectively. By defining the cost functional as

$$J = \sum_{K=\emptyset}^{N-1} e_K^T Q_{e_K} e_K \tag{2.57}$$

and using equation (2.29), the discrete-time Lyapunov function becomes

$$V_K = e_K^T \tilde{P}_K^{-1} e_K \tag{2.58}$$

where

$$\tilde{P}_{K-1}^{-1} = \tilde{A}_K^T \tilde{P}_K^{-1} \tilde{A}_K + Q_{e_K} \tag{2.59}$$

$$\tilde{A}_K = (I - K_KH_K)A_{K-1} \tag{2.60}$$

$$Q_{e_K} = \tilde{P}_K^{-1}\tilde{A}_K^{-1}(\gamma_K^{-1} + \tilde{A}_K^{-T}\tilde{P}_K^{-1}\tilde{A}_K^{-1})^{-1}\tilde{A}_K^{-T}\tilde{P}_K^{-1} \quad (2.61)$$

$$\gamma_K = (I-K_K H_K) Q_{o_K} (I-K_K H_K)^T$$

$$+ K_K R_{o_K} K_K^T \quad (2.62)$$

where Q_{e_K} is positive definite (Appendix A).

The Lyapunov function of equation (2.58) satisfies all four requirements of Section 2.2 and is therefore valid.

As in Section 2.4.1, the discrete-time filter algorithm (equations 2.50 to 2.56) can be converted to a discrete-time observer algorithm by ignoring the noise term in equation (2.52), such that the discrete measurement becomes

$$y_K = H_K x_K \quad (2.63)$$

The Lyapunov function (equation 2.58) remains the same for the discrete observer problem.

2.5 Combining Controller & Filter Lyapunov Functions for Cascaded Systems

2.5.1 Continuous, Time-Varying Combined Lyapunov Function

By including the filter in the feedback loop, and by incorporating the state estimates in the control law in the following way

$$u(t) = -L(t)\hat{x}(t) \tag{2.64}$$

the closed-loop system dynamics become

$$\dot{x}(t) = \bar{A}(t)x(t) + B(t)L(t)e(t) \tag{2.65}$$

$$\dot{e}(t) = \tilde{A}(t)e(t) \tag{2.66}$$

Proposition 2.1

$V(x,e,t)$ satisfies the sufficiency conditions for asymptotic stability in the sense of Lyapunov for the continuous, time-varying system described in equations (2.64)-(2.66) where

$$V(x,e,t) = [x^T(t) \;\; e^T(t)] \begin{bmatrix} P(t) & \emptyset \\ \emptyset & \tilde{P}^{-1}(t) \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{2.67}$$

under the assumptions that $(A(t),B(t))$ and $(A(t),\Gamma(t))$ are controllable, and $(A(t),Q_c^{1/2}(t))$ and $(A(t),H(t))$ are observable. The control gain, L(t), and the Kalman gain, K(t), are chosen as the optimal values (equations 2.16 and 2.36). Equation (2.67) comes from equations (2.21) and (2.48).

Proof:

By assuming (A(t),B(t)) and (A(t),Γ(t)) are controllable, and (A(t), $Q_c^{1/2}$(t)) and (A(t),H(t)) are observable, P(t) and $\tilde{P}^{-1}$(t) are positive definite and bounded. Thus, equation (2.67) satisfies the first three Lyapunov function requirements from Section 2.2. To evaluate the fourth condition, equation (2.67) must be differentiated.

$$\dot{V}(x,e,t) = [\dot{x}^T(t) \;\; \dot{e}^T(t)] \begin{bmatrix} P(t) & \emptyset \\ \emptyset & \tilde{P}^{-1}(t) \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}$$

$$+ \; [x^T(t) \;\; e^T(t)] \begin{bmatrix} \dot{P}(t) & \emptyset \\ \emptyset & \dot{\tilde{P}}^{-1}(t) \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} +$$

$$[x^T(t) \;\; e^T(t)] \begin{bmatrix} P(t) & \emptyset \\ \emptyset & \tilde{P}^{-1}(t) \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} \qquad (2.68)$$

From equations (2.19), (2.43), (2.65), and (2.66), the derivative of the Lyapunov function is

$$\dot{V}(x,e,t) = [x^T(t)\ \ e^T(t)] \begin{bmatrix} -L^T(t)R(t)L(t)-Q_c(t) & L^T(t)R(t)L(t) \\ L^T(t)R(t)L(t) & -\tilde{P}^{-1}(t)Q_o\tilde{P}^{-1}(t)-H^T(t)R_o^{-1}(t)H(t) \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{2.69}$$

or

$$\dot{V}(x,e,t) = [x^T(t)\ \ e^T(t)]\overline{Q}(t) \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{2.70}$$

In order to satisfy the fourth requirement (i.e. $\dot{V} \leq 0$), it is sufficient that $\overline{Q}(t)$ be negative semidefinite. However, it is not analytically possible to show that $\overline{Q}(t)$ is negative semidefinite. Through the selection of $R(t)$, $Q_c(t)$, $R_o(t)$, and $Q_o(t)$, it may be possible to show numerically that $\overline{Q}(t)$ is negative semidefinite.

A way of ensuring that $\overline{Q}(t)$ is negative semidefinite is to change G_F, Q_c and R by the following:

$$\overline{G}_F = \frac{G_F}{\alpha} \quad , \quad \overline{Q}_c = \frac{Q_c}{\alpha} \tag{2.71}$$

$$\overline{R} = \frac{R}{\alpha} \tag{2.72}$$

such that the performance index (equation 2.13) becomes

$$J = x_F^T \frac{G_F}{\alpha} x_F + \int_0^{t_F} [x^T \frac{Q_c}{\alpha} x + u^T \frac{R}{\alpha} u] dt \tag{2.73}$$

For this control problem, the control gain remains the same for all positive values of α, including $\alpha = 1$. This is so because the new Riccati matrix is a scaled function of the original one (equation 2.17) (i.e. $P_{new} = \alpha P$). The control gain is a multiple of $\bar{R}P_{new}$ which equals RP. α can be selected so that $\bar{Q}(t)$ is negative semidefinite. The controllability and observability conditions insure that $\dot{V}(x,e,t) \neq 0$ except at $x=0$ [96]; thus, the system described in equations (2.64)-(2.66) is asymptotically stable in the sense of Lyapunov, and the function

$$V = [x^T \; e^T] \begin{bmatrix} \Lambda x & 0 \\ 0 & \tilde{P}^{-1} \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \qquad (2.74)$$

which consists of combining the separate controller and filter Lyapunov functions, could be a good Lyapunov function by properly selecting α, without changing the controller or filter gains.

2.5.2 Discrete-Time Combined Lyapunov Function

For the controller/filter cascaded system, the closed-loop system difference equations are

$$x_{K+1} = \bar{A}_K x_K + B_K L_K e_K \qquad (2.75)$$

$$e_{K+1} = \tilde{A}_K e_K \tag{2.76}$$

Proposition 2.2

$V_K(x_K,e_K,t_K)$ satisfies the sufficiency conditions for asymptotic stability in the sense of Lyapunov for the discrete, time-varying system described in equations (2.75)-(2.76) where

$$V_K(x_K,e_K,t_K) = \begin{bmatrix} x_K^T & e_K^T \end{bmatrix} \begin{bmatrix} P_K & 0 \\ 0 & \tilde{P}_K^{-1} \end{bmatrix} \begin{bmatrix} x_K \\ e_K \end{bmatrix} \tag{2.77}$$

under the assumptions that (A_K,B_K) and (A_K,Γ_K) are controllable, and $(A_K,Q_{c_K}^{1/2})$ and (A_K,H_K) are observable. The control gain, L_K, and the Kalman gain, K_K, are chosen as the optimal values (equations 2.26 and 2.54).

Proof:

Equation (2.77) comes from equations (2.30) and (2.58). From the assumptions in Proposition 2.2, P_K and $\tilde{P}_K^{-1}$ are positive definite for all $K = 0,1,\ldots,N$, and bounded. Thus, the first three Lyapunov function requirements are satisfied by equation (2.77). Differencing V_K and V_{K+1}:

$$\Delta V_K = V_{K+1} - V_K \tag{2.78}$$

$$= [x_K^T \ e_K^T] \begin{bmatrix} -L_K^T R_K L_K - Q_{c_K} & L_K^T R_K L_K \\ L_K^T R_K L_K & -Q_{e_K} \end{bmatrix} \begin{bmatrix} x_K \\ e_K \end{bmatrix} \qquad (2.79)$$

or

$$\Delta V_K = [x_K^T \ e_K^T] \bar{Q}_K \begin{bmatrix} x_K \\ e_K \end{bmatrix} \qquad (2.80)$$

where Q_{e_K} is defined in equation (2.61). For $\Delta V_K \leq 0$, it is sufficient that $\bar{Q}_K$ be negative semidefinite. However, as in the continuous-time case, it is not possible to show analytically that $\bar{Q}_K$ is negative semidefinite. Through the selection of R_K, Q_{c_K}, R_{o_K}, and Q_{o_K} it is possible to show numerically that $\bar{Q}_K$ is negative semidefinite. Therefore, equation (2.77) may not be the best choice for a Lyapunov function for the cascaded system. The next logical step is to attempt to derive a Lyapunov function for the cascaded system with the technique used to derive the separate controller and filter Lyapunov functions. This is accomplished in the next section.

SECTION III

LYAPUNOV FUNCTION DERIVATION FOR CASCADED CONTROLLER/FILTER CLOSED-LOOP SYSTEM

3.1 Introduction

In the previous section, it is shown that the Lyapunov function which consisted of the sum of the separate controller and filter Lyapunov functions may not be valid for the cascaded linear, time-varying, closed-loop system. This is due to the fact that the separate Lyapunov functions are derived assuming the control law is a linear function of the true states; yet, the actual control law is a function of the estimated states from the filter.

The purpose of this section is to derive a Lyapunov function for the cascaded system, assuming the control law is a linear function of estimated states. This is accomplished by setting the Lyapunov function equal to a conditional form of the optimal return function associated with the linear-quadratic Gaussian problem. It is shown that this Lyapunov function contains the separate controller Lyapunov function, the filter Lyapunov function, and an additional quadratic term which reflects the error in the control law due to the inaccuracy of the state estimate from the filter. The Lyapunov function is derived for both the continuous-time and discrete-time problem.

3.2 Continuous, Linear, Time-Varying Systems

3.2.1 Lyapunov Function Derivation

A Lyapunov function is derived for the continuous, linear, time-varying closed-loop system by equating it to a conditional form of the optimal return function. Here, the unconditional cost function is

$$J_o = \min_{u,K} E\{J\} \tag{3.1}$$

where $E\{\cdot\}$ is the expectation operator, u is the optimal control, and K in the Kalman gain. The performance index is chosen as

$$J = x_f^T G_f x_f + \int_{\emptyset}^{t_f} (x^T(t) Q_c(t) x(t)$$

$$+ e^T(t) Q_e(t) e(t) + u^T(t) R(t) u(t))dt \tag{3.2}$$

subject to

$$\dot{x}(t) = [A(t)-B(t)L(t)]x(t)+B(t)L(t)e(t)+\Gamma(t)\omega(t) \quad (3.3)$$

$$\dot{e}(t) = [A(t)-K(t)H(t)]e(t)-K(t)V(t)+\Gamma(t)\omega(t) \quad (3.4)$$

where

$$\omega(t) \sim N(\emptyset,\bar{Q}(t)\delta(t-\tau)) \ , \ V(t) \sim N(\emptyset,\bar{R}(t)\delta(t-\tau)) \quad (3.5)$$

$$u(t) = -L(t)[x(t) - e(t)] \quad (3.6)$$

G_f, $Q_c(t)$, $Q_e(t)$, and $\bar{Q}(t)$ are symmetric positive semidefinite, and $R(t)$ and $\bar{R}(t)$ are symmetric positive definite.

It must be noted that the Lyapunov functions are derived using the noise properties of the system states and measurements as design parameters, rather than actual power spectral densities. For this reason, the Lyapunov functions are valid only for the closed-loop system with an observer in the loop.

First, rewrite the performance index in terms of x(t) and e(t) only and take the expectation.

$$E\{J\} = E\{[x_f^T \ e_f^T]\begin{bmatrix} G_f & \emptyset \\ \emptyset & \emptyset \end{bmatrix}\begin{bmatrix} x_f \\ e_f \end{bmatrix}$$

$$+ \int_{\emptyset}^{t_f}[x^T(t) \ \ e^T(t)]\begin{bmatrix} L^T(t)R(t)L(t)+Q_c(t) & -L^T(t)R(t)L(t) \\ -L^T(t)R(t)L(t) & L^T(t)R(t)L(t)+Q_e(t) \end{bmatrix}\begin{bmatrix} x(t) \\ e(t) \end{bmatrix}dt\} \qquad (3.7)$$

Carrying through the expectation, equation (3.7) becomes

$$E\{J\} = \mathrm{tr}\begin{bmatrix} G_f & \emptyset \\ \emptyset & \emptyset \end{bmatrix}\begin{bmatrix} X_f & S_f \\ S_f^T & P_f \end{bmatrix}$$

$$+ \int_{\emptyset}^{t_f} \mathrm{tr}\begin{bmatrix} L^T(t)R(t)L(t)+Q_c(t) & -L^T(t)R(t)L(t) \\ -L^T(t)R(t)L(t) & L^T(t)R(t)L(t)+Q_e(t) \end{bmatrix}$$

$$* \begin{bmatrix} X(t) & S(t) \\ S^T(t) & P(t) \end{bmatrix}dt \qquad (3.8)$$

where tr[•] denotes the trace and

$$X(t) = E[x(t)x^T(t)] \ , \ S(t) = E[x(t)e^T(t)]$$

$$P(t) = E[e(t)e^T(t)] \qquad (3.9)$$

The next step is to augment the dynamic constraint equations (3.3 and 3.4) to the performance index. To do this, these equations have to be translated to dynamic constraints in X(t), S(t), and P(t). These constraints are derived in Appendix B and can be written as

$$\begin{bmatrix} \dot{X}(t) & \dot{S}(t) \\ \dot{S}^T(t) & \dot{P}(t) \end{bmatrix} = \begin{bmatrix} \bar{A}(t) & B(t)L(t) \\ 0 & \tilde{A}(t) \end{bmatrix} \begin{bmatrix} X(t) & S(t) \\ S^T(t) & P(t) \end{bmatrix}$$

$$+ \begin{bmatrix} X(t) & S(t) \\ S^T(t) & P(t) \end{bmatrix} \begin{bmatrix} \bar{A}^T(t) & 0 \\ L^T(t)B^T(t) & \tilde{A}^T(t) \end{bmatrix}$$

$$+ \begin{bmatrix} \bar{Q}(t) & \bar{Q}(t) \\ \bar{Q}(t) & K(t)\bar{R}(t)K^T(t)+\bar{Q}(t) \end{bmatrix} \qquad (3.10)$$

where

$$\bar{A}(t) = A(t) - B(t)L(t) \qquad (3.11)$$

$$\tilde{A}(t) = A(t) - K(t)H(t) \qquad (3.12)$$

Using the following definitions

$$\hat{X}(t) \triangleq \begin{bmatrix} X(t) & S(t) \\ S^T(t) & P(t) \end{bmatrix} \qquad (3.13)$$

$$\hat{A}(t) \triangleq \begin{bmatrix} \bar{A}(t) & B(t)L(t) \\ 0 & \tilde{A}(t) \end{bmatrix} \qquad (3.14)$$

$$\hat{Q}(t) \triangleq \begin{bmatrix} L^T(t)R(t)L(t)+Q_c(t) & -L^T(t)R(t)L(t) \\ -L^T(t)R(t)L(t) & L^T(t)R(t)L(t)+Q_e(t) \end{bmatrix} \quad (3.15)$$

$$\hat{\hat{Q}}(t) \triangleq \begin{bmatrix} \bar{Q}(t) & \bar{Q}(t) \\ \bar{Q}(t) & K(t)\bar{R}(t)K^T(t)+\bar{Q}(t) \end{bmatrix} \quad (3.16)$$

the augmented performance index becomes

$$J' = trG_fX_f + \int_{\emptyset}^{t_f} tr\hat{Q}(t)\hat{X}(t)$$

$$+ \; tr \wedge(t)[\hat{A}(t)\hat{X}(t) + \hat{X}(t)\hat{A}^T(t) + \hat{\hat{Q}}(t) - \dot{\hat{X}}(t)]dt \quad (3.17)$$

where $\wedge(t)$ is the Lagrange multiplier defined by the following

$$\wedge(t) \triangleq \begin{bmatrix} \wedge_X(t) & \wedge_S(t) \\ \wedge_S^T(t) & \wedge_P(t) \end{bmatrix} \quad (3.18)$$

The cost function can be manipulated into the form [26]

$$J' = tr\hat{X}(t_\emptyset)\wedge(t_\emptyset) + \int_{t_\emptyset}^{t_f} tr(\wedge(t)Q(t))d\tau \quad (3.19)$$

$$= E\{[x^T(t) \;\; e^T(t)]\wedge(t)\begin{bmatrix} x(t) \\ e(t) \end{bmatrix}\} + \int_{t_\emptyset}^{t_f} tr(\wedge(t)Q(t))d\tau \quad (3.20)$$

The first term in equation (3.20) represents the deterministic part of the problem and is an expected value of a quadratic term in the states x and e. This term will be called the Lyapunov function and is of the form

$$V(x,e,t) = [x^T(t) \ \ e^T(t)] \begin{bmatrix} \Lambda_X(t) & \Lambda_S(t) \\ \Lambda_S^T(t) & \Lambda_P(t) \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \quad (3.21)$$

The Hamiltonian of the system is

$$H(t) = \mathrm{tr}\hat{Q}(t)\hat{X}(t)$$

$$+ \mathrm{tr}\,\Lambda(t)[\hat{A}(t)\hat{X}(t) + \hat{X}(t)\hat{A}^T(t) + \hat{\hat{Q}}(t)] \quad (3.22)$$

From the Euler-Lagrange equations,

$$\dot{\Lambda}(t) = -H_{\hat{X}} = -\Lambda(t)\hat{A}(t) - \hat{A}(t)\Lambda(t) - \hat{Q}(t) \quad (3.23)$$

$$\Lambda(t_f) = \begin{bmatrix} G_f & 0 \\ 0 & 0 \end{bmatrix} \quad (3.24)$$

or

$$\begin{bmatrix} \dot{\Lambda}_X(t) & \dot{\Lambda}_S(t) \\ \dot{\Lambda}_S^T(t) & \dot{\Lambda}_P(t) \end{bmatrix} = -\begin{bmatrix} \Lambda_X(t) & \Lambda_S(t) \\ \Lambda_S^T(t) & \Lambda_P(t) \end{bmatrix} \begin{bmatrix} \bar{A}(t) & B(t)L(t) \\ 0 & \tilde{A}(t) \end{bmatrix}$$

$$- \begin{bmatrix} \bar{A}^T(t) & 0 \\ L^T(t)B^T(t) & \tilde{A}^T(t) \end{bmatrix} \begin{bmatrix} \Lambda_X(t) & \Lambda_S(t) \\ \Lambda_S^T(t) & \Lambda_P(t) \end{bmatrix}$$

$$- \begin{bmatrix} L^T(t)R(t)L(t)+Q_c(t) & -L^T R(t)L(t) \\ -L^T(t)R(t)L(t) & L^T(t)R(t)L(t)+Q_e(t) \end{bmatrix} \quad (3.25)$$

$$\begin{bmatrix} \Lambda_X(t_f) & \Lambda_S(t_f) \\ \Lambda_S^T(t_f) & \Lambda_P(t_f) \end{bmatrix} = \begin{bmatrix} G_f & 0 \\ 0 & 0 \end{bmatrix} \quad (3.26)$$

From equation (3.25)

$$\dot{\Lambda}_X(t) = -\Lambda_X(t)\bar{A}(t) - \bar{A}^T(t)\Lambda_X(t)$$

$$- L^T(t)R(t)L(t) - Q_c(t) \quad , \Lambda_X(t_f)=G_f \quad (3.27)$$

$$\dot{\Lambda}_S(t) = -\Lambda_S(t)\tilde{A}(t) - \bar{A}^T(t)\Lambda_S(t)$$

$$- \Lambda_X(t)B(t)L(t) + L^T(t)R(t)L(t) \quad , \Lambda_S(t_f)=0 \quad (3.28)$$

$\Lambda_X(t)$ is equivalent to the controller Riccati matrix which implies that the control gain $L(t)$ is

$$L(t) = R^{-1}(t)B^T(t)\Lambda_x(t) \tag{3.29}$$

For $\Lambda_S(t_f) = 0$, $\Lambda_S(t) = 0$ for $0 \leq t \leq t_f$. Therefore, the Lyapunov function reduces to

$$V(x,e,t) = [x^T(t) \;\; e^T(t)] \begin{bmatrix} \Lambda_x(t) & 0 \\ 0 & \Lambda_p(t) \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{3.30}$$

where $\Lambda_x(t)$ is defined by equation (3.27) and

$$\dot{\Lambda}_p(t) = -\Lambda_p(t)\tilde{A}(t) - \tilde{A}^T(t)\Lambda_p(t)$$

$$- L^T(t)R(t)L(t) - Q_e(t) \tag{3.31}$$

$$\Lambda_p(t_f) = 0 \tag{3.32}$$

By defining

$$\Lambda_p(t) = \tilde{P}^{-1}(t) + P_e(t) \tag{3.33}$$

equation (3.31) can be split into two differential equations

$$\dot{\tilde{P}}(t) = -\tilde{P}^{-1}(t)\tilde{A}(t) - \tilde{A}^T(t)\tilde{P}^{-1}(t)$$

$$- Q_e(t) \;,\; \tilde{P}^{-1}(t_f) = \tilde{P}_f^{-1} \tag{3.34}$$

$$\dot{P}_e(t) = -P_e(t)\tilde{A}(t) - \tilde{A}^T(t)P_e(t)$$

$$- L^T(t)R(t)L(t) \quad , \quad P_e(t_f) = - \tilde{P}_f^{-1} \qquad (3.35)$$

The Lyapunov function becomes

$$V(x,e,t) = x^T(t)\Lambda_x(t)x(t)$$

$$+ e^T(t)\tilde{P}^{-1}(t)e(t) + e^T(t)P_e(t)e(t) \qquad (3.36)$$

where the first quadratic term is the controller Lyapunov function, the second term is the filter Lyapunov function, and the third term is an additional term that reflects the error in the control law due to the inaccuracy of the state estimate from the filter. As the error increases, this term has a stronger influence on the Lyapunov function. Note that $P_e(t)$ (equation 3.35) is affected by both the Kalman gain, K(t), in $\tilde{A}(t)$, and the control law gain, L(t). The next step is to determine if equation (3.30) is a valid Lyapunov function.

3.2.2 Lyapunov Function Validation

Proposition 3.1

V(x,e,t) from equation (3.30) satisfies the sufficiency conditions for asymptotic stability in the sense of Lyapunov for the continuous, time-varying system described in equations (2.64)-(2.66) under the assumptions that (A(t),B(t)) and (A(t),Γ(t)) are controllable and (A(t),H(t)) and (A(t),$Q_c^{1/2}$(t)) are observable [121]. The control gain, L(t), and the Kalman gain, K(t), are chosen as their optimal values (equations 3.29 and 2.36).

Proof:

By assuming (A(t),B(t)) and (A(t),Γ(t)) are controllable and (A(t),H(t)) and (A(t),$Q_c^{1/2}$(t))) are observable, $\Lambda_X(t)$, which is the control Riccati matrix, and $\tilde{P}^{-1}(t)$, which is the observability Grammian matrix, are positive definite for $t>t_0$ and bounded. With $\tilde{P}^{-1}(t)$ positive definite and $P_e(t)$ positive semidefinite [25], $\Lambda_P(t)$ is positive definite from equation (3.33). This implies

$$V(x,e,t) > 0 \tag{3.37}$$

for $t \in [0,t_f)$ and all x(t) and e(t) not equal to zero.

Therefore, the first three Lyapunov function requirements of Section 2.2 are satisfied. The derivative of equation (3.30) is

$$\dot{V}(x,e,t) = [x^T(t) \;\; e^T(t)]$$

$$* \begin{bmatrix} -L^T(t)R(t)L(t)-Q_c(t) & L^T(t)R(t)L(t) \\ L^T(t)R(t)L(t) & -L^T(t)R(t)L(t)-Q_e(t) \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}$$

$$\triangleq [x^T(t) \;\; e^T(t)]Q'(t)\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \qquad (3.38)$$

Rewriting $Q'(t)$ as

$$Q'(t) = -\begin{bmatrix} L^T(t) \\ -L^T(t) \end{bmatrix}[R(t)][L(t) \;\; -L(t)]$$

$$+ \begin{bmatrix} -Q_c(t) & 0 \\ 0 & -Q_e(t) \end{bmatrix} \qquad (3.39)$$

it is obvious that for $R(t)>0$, the first quadratic matrix in equation (3.39) is negative semidefinite, and for $Q_c(t)\geq 0$ and $Q_e(t)\geq 0$, the second matrix in equation (3.39) is negative semidefinite. In addition, the controllability and observability conditions assure that $\dot{V}(x,e,t)=0$ only when $x=0$. Therefore, the derivative of the Lyapunov function becomes

$$\dot{V}(x,e,t) < 0 \qquad (3.40)$$

for $x \neq 0$, which satisfies the fourth requirement for a Lyapunov function. The controllability and observability conditions insure that $\dot{V}(x,e,t) \neq 0$ except at $x=0$ [96]; thus, the system described in equations (2.64)-(2.65) is asymptotically stable in the sense of Lyapunov. Therefore, equation (3.30) represents a good Lyapunov function for the cascaded, linear, time-varying, closed-loop system. The same Lyapunov funtion (equation 3.30) is derived using the Hamilton-Jacobi equation (Appendix C).

It is important to note that for the linear homogeneous system (equation 3.3), the adjoint equation propogates the solution of the original equation backward in time [25]; thus, acting as a predictor of the system. With the Lyapunov function defined as a conditional form of the optimal return function (equation 3.30), the Lyapunov matrix, defined by the backward Riccati differential equation (3.25), has predictive qualities for the value of the cost.

3.2.3 Extension to Nonlinear Systems

Based on the results by Song and Speyer [119,120], this Lyapunov function can be extended to the class of nonlinear systems where the system dynamics are linear and the filter measurement equations are nonlinear functions of the states. The nonlinear measurements can be transformed into a linear function of the states where the coefficient matrix is a nonlinear function of the observations. The pseudomeasurement observer (PMO) and the modified gain extended Kalman observer (MGEKO) are based on this concept. The algorithm for the PMO is presented in detail in Section 6.4.1.

The Lyapunov function (equation 3.36) is changed by replacing $P_e(t)$ with the inverse of the observability Grammian of the PMO (Section 6.4.1). The observability Grammian matrix of the PMO is positive definite for $t \geq t_0$ [103]. The Lyapunov functions derived in this dissertation are valid for this class of nonlinear systems.

3.3 Discrete-Time Linear System

3.3.1 Lyapunov Function Derivation

A Lyapunov function is derived for the discrete-time, linear closed-loop system by equating it to a conditional form of the optimal return function. Here, the unconditional cost function is

$$J_o = \min_{u_K, K_K} E\{J\} \tag{3.41}$$

where $E\{\bullet\}$ is the expectation operator, u_K is the optimal control, and K_K is the Kalman gain. The performance index is chosen as

$$J = x_N^T G_N x_N$$

$$+ \sum_{K=0}^{N-1} x_K^T Q_{c_K} x_K + e_K^T Q_{e_K} e_K + u_K^T R_K u_K \qquad (3.42)$$

subject to

$$x_{K+1} = (A_K - B_K L_K) x_K + B_K L_K e_K + \Gamma_K \omega_K \qquad (3.43)$$

$$e_{K+1} = (A_K - K_{K+1} H_{K+1} A_K) e_K - K_{K+1} \mathcal{V}_{K+1} + \Gamma_K \omega_K \qquad (3.44)$$

$$\omega_K \sim N(0, \overline{Q}_K) \quad , \quad \mathcal{V}_K \sim N(0, \overline{R}_K) \qquad (3.45)$$

$$u_K = - L_K (x_K - e_K) \qquad (3.46)$$

G_N , Q_{c_K} , and $\overline{Q}_K$ are symmetric positive semidefinite, and R_K and $\overline{R}_K$ are symmetric positive definite. Rewriting the performance index in terms of x_K and e_K and taking the expectation gives

$$E\{J\} = E\{[x_N^T \; e_N^T] \begin{vmatrix} G_N & \emptyset \\ \emptyset & \emptyset \end{vmatrix} \begin{vmatrix} x_N \\ e_N \end{vmatrix}$$

$$+ \sum_{K=\emptyset}^{N-1} [x_K^T \; e_K^T] \begin{vmatrix} L_K^T R_K L_K + Q_{c_K} & -L_K^T R_K L_K \\ -L_K^T R_K L_K & L_K^T R_K L_K + Q_{e_K} \end{vmatrix} \begin{vmatrix} x_K \\ e_K \end{vmatrix}\} \quad (3.47)$$

Carrying through the expectation, equation (3.47) becomes

$$E\{J\} = \mathrm{tr} \begin{bmatrix} G_N & \emptyset \\ \emptyset & \emptyset \end{bmatrix} \begin{bmatrix} X_N & S_N \\ S_N^T & P_N \end{bmatrix}$$

$$+ \sum_{K=1}^{N-1} \mathrm{tr} \begin{bmatrix} L_K^T R_K L_K + Q_{c_K} & -L_K^T R_K L_K \\ -L_K^T R_K L_K & L_K^T R_K L_K + Q_{e_K} \end{bmatrix} \begin{bmatrix} X_K & S_K \\ S_K^T & P_K \end{bmatrix} \quad (3.48)$$

where

$$X_K \triangleq E[x_K x_K^T] \quad , \quad S_K \triangleq E[x_K e_K^T] \quad , \quad P_K \triangleq E[e_K e_K^T] \quad (3.49)$$

To augment the dynamic constraint equations to the performance index, equations (3.43) and (3.44) have to be translated to difference equations in X_K, S_K, and P_K. These difference equations are derived in Appendix B. The result is

$$\begin{bmatrix} X_{K+1} & S_{K+1} \\ S_{K+1}^T & P_{K+1} \end{bmatrix} = \begin{bmatrix} \bar{A}_K & B_K L_K \\ \emptyset & \tilde{A}_K \end{bmatrix} \begin{bmatrix} X_K & S_K \\ S_K^T & P_K \end{bmatrix} \begin{bmatrix} \bar{A}_K^T & \emptyset \\ L_K^T B_K^T & \tilde{A}_K^T \end{bmatrix}$$

$$+ \begin{bmatrix} \bar{Q}_K & \bar{Q}_K \\ \bar{Q}_K & K_K R_K K_K^T + \bar{Q}_K \end{bmatrix} \tag{3.50}$$

where

$$\bar{A}_K = A_K - B_K L_K \tag{3.51}$$

$$\tilde{A}_K = A_K - K_{K+1} H_{K+1} A_K \tag{3.52}$$

The augmented performance index is

$$J' = \operatorname{tr} G_N X_N + \sum_{K=\emptyset}^{N-1} \operatorname{tr} \hat{Q}_K \hat{X}_K$$

$$+ \operatorname{tr} \Lambda_{K+1} [\hat{A}_K X_K \hat{A}_K^T + \hat{\hat{Q}}_K - \hat{X}_{K+1}] \tag{3.53}$$

where

$$\hat{X} = \begin{bmatrix} X_K & S_K \\ S_K^T & P_K \end{bmatrix} \tag{3.54}$$

$$\hat{A}_K = \begin{bmatrix} \bar{A}_K & B_K L_K \\ \emptyset & \tilde{A}_K \end{bmatrix} \tag{3.55}$$

$$\hat{Q}_K = \begin{bmatrix} L_K^T R_K L_K + Q_{c_K} & -L_K^T R_K L_K \\ -L_K^T R_K L_K & L_K^T R_K L_K + Q_{e_K} \end{bmatrix} \tag{3.56}$$

$$\hat{\hat{Q}}_K = \begin{bmatrix} \bar{Q}_K & \bar{Q}_K \\ \bar{Q}_K & K_K R_K K_K^T + \bar{Q}_K \end{bmatrix} \tag{3.57}$$

$$\Lambda_K = \begin{bmatrix} \Lambda_{x_K} & \Lambda_{s_K} \\ \Lambda_{s_K}^T & \Lambda_{p_K} \end{bmatrix} \tag{3.58}$$

and Λ_K is the Lagrange multiplier. The cost function can be manipulated into the form [26]

$$J' = \operatorname{tr} \hat{X}_o \Lambda_o + \sum_{i=1}^{N-1} \operatorname{tr} \Lambda_i Q_i$$

$$= E[V_K(x_K, e_K, t_K)] + \sum_{i=1}^{N-1} \operatorname{tr} \Lambda_i Q_i \tag{3.59}$$

where $V_K(x_K, e_K, t_K)$ is chosen to be a Lyapunov function of the form

$$V_K(x_K, e_K, t_K) = [x_K^T \; e_K^T] \begin{bmatrix} \Lambda_{x_K} & \Lambda_{s_K} \\ \Lambda_{s_K}^T & \Lambda_{p_K} \end{bmatrix} \begin{bmatrix} x_K \\ e_K \end{bmatrix} \tag{3.60}$$

The discrete Hamiltonian of the system is

$$H_K = \operatorname{tr} \hat{Q}_K \hat{X}_K + \operatorname{tr} \Lambda_{K+1} [\hat{A}_K \hat{X}_K \hat{A}_K^T + \hat{\hat{Q}}_K] \tag{3.61}$$

From the Euler-Lagrange equations

$$\Lambda_K = H_{\hat{X}_K} = \hat{A}_K^T \Lambda_{K+1} \hat{A}_K + \hat{Q}_K \quad , \quad \Lambda_N = G_N \qquad (3.62)$$

or

$$\begin{bmatrix} \Lambda_{X_K} & \Lambda_{S_K} \\ \Lambda_{S_K}^T & \Lambda_{P_K} \end{bmatrix} = \begin{bmatrix} \bar{A}_K^T & \emptyset \\ L_K^T B_K^T & \tilde{A}_K^T \end{bmatrix} \begin{bmatrix} \Lambda_{X_{K+1}} & \Lambda_{S_{K+1}} \\ \Lambda_{S_{K+1}}^T & \Lambda_{P_{K+1}} \end{bmatrix} \begin{bmatrix} \bar{A}_K & B_K L_K \\ \emptyset & \tilde{A}_K \end{bmatrix}$$

$$+ \begin{bmatrix} L_K^T R_K L_K + Q_{c_K} & -L_K^T R_K L_K \\ -L_K^T R_K L_K & L_K^T R_K L_K + Q_{e_K} \end{bmatrix} \qquad (3.63)$$

$$\begin{bmatrix} \Lambda_{X_N} & \Lambda_{S_N} \\ \Lambda_S^{T_N} & \Lambda_{P_N} \end{bmatrix} = \begin{bmatrix} G_N & \emptyset \\ \emptyset & \emptyset \end{bmatrix} \qquad (3.64)$$

for $K = N-1,\ldots,1$.

From the difference equation for Λ_{X_K} ,

$$\Lambda_{X_K} = \bar{A}_K^T \Lambda_{X_{K+1}} \bar{A}_K + L_K^T R_K L_K + Q_K \quad , \quad \Lambda_{X_N} = G_N \quad (3.65)$$

it is obvious that Λ_{X_K} is equal to the controller Riccati matrix, which implies that the discrete control gain, L_K, is

$$L_K = (R_K + B_K^T \Lambda_{X_{K+1}} B_K)^{-1} B_K^T \Lambda_{X_{K+1}} A_K \qquad (3.66)$$

From equation (3.62), the difference equation for Λ_{S_K} is

$$\Lambda_{S_K} = \bar{A}_K^T \Lambda_{S_{K+1}} \tilde{A}_K + \bar{A}_K^T \Lambda_{X_{K+1}} B_K L_K - L_K^T R_K L_K \;,\; \Lambda_{S_N} = 0 \quad (3.67)$$

With L_K defined in equation (3.66) and $\Lambda_{S_N} = 0$,

$$\Lambda_{S_K} = 0 \quad , \quad K = N, \ldots, 1 \qquad (3.68)$$

Substituting equation (3.68) into (3.63), the difference equation for Λ_{P_K} is

$$\Lambda_{P_K} = \tilde{A}_K^T \Lambda_{P_{K+1}} \tilde{A}_K + L_K^T (B_K^T \Lambda_{X_{K+1}} B_K + R_K) L_K$$

$$+ Q_{e_K} \;,\; \Lambda_{P_N} = 0 \qquad (3.69)$$

By defining

$$\Lambda_{P_K} = \tilde{P}_K^{-1} + P_{e_K} \qquad (3.70)$$

equation (3.69) can be split into two difference equations

$$\tilde{P}_K^{-1} = \tilde{A}_K^T \tilde{P}_{K+1}^{-1} \tilde{A}_K + Q_{e_K} \quad , \quad \tilde{P}_N^{-1} = \tilde{P}_N^{-1} \qquad (3.71)$$

$$P_{e_K} = \tilde{A}_K^T P_{e_{K+1}} \tilde{A}_K + L_K^T (B_K^T \Lambda_{X_{K+1}} B_K + R_K) L_K \ , \ P_{e_N} = -\tilde{P}_N^{-1} \quad (3.72)$$

The Lyapunov function becomes

$$V_K(x_K, e_K, t_K) = [x_K^T \ e_K^T] \begin{bmatrix} \Lambda_{X_K} & 0 \\ 0 & \Lambda_{P_K} \end{bmatrix} \begin{bmatrix} x_K \\ e_K \end{bmatrix} \quad (3.73)$$

or

$$V_K = x_K^T \Lambda_{X_K} x_K + e_K^T \tilde{P}_K^{-1} e_K + e_K^T P_{e_K} e_K \quad (3.74)$$

where Λ_{X_K}, $\tilde{P}_K^{-1}$, and P_{e_K} are defined by equations (3.65), (3.71), and (3.72), respectively. The first quadratic term is the discrete controller Lyapunov function, the second term is the discrete observer Lyapunov function, and the third term is an additional term that, like in the continuous-time case, reflects the error in the control law due to the inaccuracy of the state estimate from the filter. Note that P_{e_K} (equation 3.72) is affected by both the Kalman gain, K_K, in $\tilde{A}_K$, and the control law gain, L_K. The question arises as to whether equation (3.74) is a valid Lyapunov function for the linear, discrete-time, closed-loop system.

3.3.2 Lyapunov Function Validation

Proposition 3.2

$V_K(x_K,e_K,t_K)$ from equation (3.73) satisfies the sufficiency conditions for asymptotic stability in the sense of Lyapunov for the discrete, time-varying system described in equations (2.76)-(2.77) under the assumptions that (A_K,B_K) and (A_K,Γ_K) are controllable and (A_K,H_K) and $(A_K,Q_{c_K}^{1/2})$ are observable. The control gain, L_K, and the Kalman gain, K_K, are chosen as the optimal values (equations 2.26 and 2.54).

Proof:

By assuming (A_K,B_K) and (A_K,Γ_K) are controllable and (A_K,H_K) and $(A_K,Q_{c_K}^{1/2})$ are observable, Λ_{X_K}, which is the control Riccati matrix (equation 3.65), and $\tilde{P}_K^{-1}$, which is the observability Grammian matrix (equation 3.71), are positive definite for $K>0$ and bounded. With $\tilde{P}_K^{-1}$ positive definite and P_{e_K} positive semidefinite [25], Λ_{P_K} is positive definite from equation (3.70). This implies

$$V_K(x_K,e_K,t_K) > 0 \tag{3.75}$$

for $t_K = N,\ldots,1$ and all x_K and e_K not equal to zero.

Therefore, the first three Lyapunov function requirements of Section 2.2 are satisfied. Looking at the following difference equation

$$\Delta V_K = V_{K+1} - V_K \tag{3.76}$$

$$= [x_K^T \; e_K^T] \begin{bmatrix} -L_K^T R_K L_K - Q_{c_K} & L_K^T R_K L_K \\ L_K^T R_K L_K & -L_K^T R_K L_K - Q_{e_K} \end{bmatrix} \begin{bmatrix} x_K \\ e_K \end{bmatrix}$$

$$\equiv [x_K^T \; e_K^T] Q_K' \begin{bmatrix} x_K \\ e_K \end{bmatrix} \tag{3.77}$$

Rewriting Q_K' as

$$Q_K' = -\begin{bmatrix} L_K^T \\ -L_K^T \end{bmatrix} R_K [L_K \; -L_K] + \begin{bmatrix} -Q_{c_K} & 0 \\ 0 & -Q_{e_K} \end{bmatrix} \tag{3.78}$$

it is obvious that for $R_K > 0$, the first quadratic matrix in (3.78) is negative semidefinite, and for $Q_{c_K} \geq 0$ and $Q_{e_K} \geq 0$, the second matrix in (3.78) is negative semidefinite. In addition, the controllability and observability conditions assure that $\Delta V_K = 0$ only when $x_K = 0$. Therefore,

$$\Delta V_K(x_K, e_K, t_K) < 0 \quad , \quad K = N, \ldots, 1 \tag{3.79}$$

for $x_K \neq 0$, which satisfies the fourth requirement of a Lyapunov function. The controllability and observability conditions insure that $\Delta V_K(x_K, e_K, t_K) \neq 0$ except at $x_K = 0$; thus, the system described by equations (2.76)-(2.77) is asymptotically stable in the sense of Lyapunov [121]. Therefore, equation (3.73) represents a good Lyapunov function for the discrete, cascaded, linear, closed-loop system.

SECTION IV

LYAPUNOV STABILITY FOR PARAMETER VARIATIONS

4.1 Introduction

The design of both control laws and observers is dependent, to some degree, on the knowledge of the true systems dynamics and measurement parameters. In most cases, they are both designed with the assumption that the dynamic and measurement parameters are known exactly. In most real-world problems, this assumption is not always valid and could cause some stability problems if the parameter variations are significant. In some cases the parameter uncertainties can be modelled as random variables or constants, and incorporated into the estimation algorithm. The parameter uncertainties become additional state variables to be estimated by the filter, but this increases the computational load on the algorithm [66].

For linear, time-invariant systems, eigenvalue analysis is very useful for identifying acceptable bounds of parameter variations under which the system remains stable [122]. The purpose of this section is to develop the means of analyzing the stability of time-varying systems through the use of Lyapunov's second method. The first step is to expand the Lyapunov functions derived in Section III, to account for parameter variations. Then, as in Section III, a Lyapunov function is derived to account for parameter variations in the dynamic and measurement models.

The first step in this section is to identify these parameter uncertainties and incorporate them into the linear, time-varying, cascaded closed-loop system model. Then, as before, the linear-quadratic-Gaussian optimization problem is solved, subject to these new dynamic constraint equations. From this optimization problem, a new Lyapunov function is derived which considers variations in the dynamic and measurement models. This derivation is done for both the continuous-time and discrete-time systems.

4.2 Continuous, Linear, Time-Varying Systems

4.2.1 Extension of Lyapunov Function From Section 3.2.1

The Lyapunov function for the continuous, linear, time-varying system without parameter uncertainties is rewritten as

$$V(t) = [x^T(t)\ \ e^T(t)]\begin{bmatrix}\Lambda_x(t) & 0 \\ 0 & \Lambda_p(t)\end{bmatrix}\begin{bmatrix}x(t) \\ e(t)\end{bmatrix} \tag{4.1}$$

where

$$\dot{\Lambda}_x(t) = -\Lambda_x(t)\bar{A}(t) - \bar{A}^T(t)\Lambda_x(t)$$

$$- L^T(t)R(t)L(t) - Q_c(t) \tag{4.2}$$

$$\Lambda_x(t_f) = G_f \tag{4.3}$$

$$\dot{\Lambda}_p(t) = -\Lambda_p(t)\tilde{A}(t) - \tilde{A}^T(t)\Lambda_p(t)$$

$$- L^T(t)R(t)L(t) - Q_e(t) \tag{4.4}$$

$$\Lambda_p(t_f) = 0 \tag{4.5}$$

and the closed-loop system dynamics are

$$\dot{x}(t) = (A(t)-B(t)L_c(t))x(t) + B(t)L_c(t)e(t) \tag{4.6}$$

$$\dot{\hat{x}}(t) = (A_c(t)-B_c(t)L_c(t))\hat{x}(t)$$

$$+ K_c(t)[y(t)-H_c(t)\hat{x}(t)+M_c(t)L_c(t)\hat{x}(t)] \tag{4.7}$$

$$y(t) = H(t)x(t) - M(t)L_c(t)\hat{x}(t) \tag{4.8}$$

where A, B, H, and M are the unknown true system parameters, and A_c, B_c, H_c, M_c, K_c, and L_c are the designed (or nominal) system parameters. Define the modelling errors (or deviations from the nominal) as

$$\triangle A = A - A_c \tag{4.9}$$

$$\triangle B = B - B_c \tag{4.10}$$

$$\triangle H = H - H_c \tag{4.11}$$

$$\triangle M = M - M_c \tag{4.12}$$

The system dynamics can be rewritten to emphasize the modelling errors in the following way

$$\dot{x}(t) = (A_c(t) - B_c(t)L_c(t) + D_{AB}(t))x(t)$$

$$+ (B_c(t)L_c(t) + D_B(t))e(t) \tag{4.13}$$

where

$$D_{AB}(t) = (A(t) - A_c(t)) - (B(t) - B_c(t))L_c(t) \tag{4.14}$$

$$D_B(t) = (B(t) - B_c(t))L_c(t) \tag{4.15}$$

The estimation error, defined as

$$e(t) = x(t) - \hat{x}(t) \tag{4.16}$$

has the following dynamics

$$\dot{e}(t) = D(t)x(t) + (A_c(t)-K_c(t)H_c(t) + D_{BM}(t))e(t) \tag{4.17}$$

where

$$D(t) = (A(t)-A_c(t)) - K_c(t)(H(t)-H_c(t))$$

$$- (B(t)-B_c(t))L_c(t) + K_c(t)(M(t)-M_c(t))L_c(t) \tag{4.18}$$

$$D_{BM}(t) = (B(t)-B_c(t))L_c(t)-K_c(t)(M(t)-M_c(t))L_c(t) \tag{4.19}$$

Therefore, the closed-loop system is

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A_c(t)-B_c(t)L_c(t)+D_{AB}(t)) & (B_c(t)L_c(t)+D_B(t)) \\ D(t) & (A_c(t)-K_c(t)H_c(t)+D_{BM}(t)) \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{4.20}$$

Equations (4.2) and (4.4) can be rewritten as

$$\begin{bmatrix} \dot{\Lambda}_X(t) & 0 \\ 0 & \dot{\Lambda}_P(t) \end{bmatrix} + \begin{bmatrix} \Lambda_X(t) & 0 \\ 0 & \Lambda_P(t) \end{bmatrix} \begin{bmatrix} \bar{A}(t) & B(t)L(t) \\ 0 & \tilde{A}(t) \end{bmatrix}$$

$$+ \begin{bmatrix} \bar{A}^T(t) & 0 \\ L^T(t)B^T(t) & \tilde{A}^T(t) \end{bmatrix} \begin{bmatrix} \Lambda_X(t) & 0 \\ 0 & \Lambda_P(t) \end{bmatrix}$$

$$= -\begin{bmatrix} L^T(t)R(t)L(t)+Q_c(t) & -L^T(t)R(t)L(t) \\ -L^T(t)R(t)L(t) & L^T(t)R(t)L(t)+Q_e(t) \end{bmatrix} \quad (4.21)$$

Incorporating the effects of the modelling errors into this equation results in

$$\begin{bmatrix} \dot{\Lambda}_X(t) & 0 \\ 0 & \dot{\Lambda}_P(t) \end{bmatrix} + \begin{bmatrix} \Lambda_X(t) & 0 \\ 0 & \Lambda_P(t) \end{bmatrix} \begin{bmatrix} \bar{A}'(t) & B'(t) \\ D(t) & \tilde{A}'(t) \end{bmatrix}$$

$$+ \begin{bmatrix} \bar{A}'^T(t) & D^T(t) \\ B'^T(t) & \tilde{A}'^T(t) \end{bmatrix} \begin{bmatrix} \Lambda_X(t) & 0 \\ 0 & \Lambda_P(t) \end{bmatrix}$$

$$= -\begin{bmatrix} L_c^T(t)R(t)L_c(t)+Q_c(t) & -L_c^T(t)R(t)L_c(t) \\ -L_c^T(t)R(t)L_c(t) & L_c^T(t)R(t)L_c(t)+Q_e(t) \end{bmatrix}$$

$$+ \begin{bmatrix} \Lambda_X(t)D_{AB}+D_{AB}^T\Lambda_X(t) & \Lambda_X(t)D_B+D^T\Lambda_P(t) \\ \Lambda_P(t)D+D_B^T\Lambda_X(t) & \Lambda_P(t)D_{BM}+D_{BM}^T\Lambda_P(t) \end{bmatrix} \quad (4.22)$$

where

$$\bar{A}'(t) = A_c(t) - B_c(t)L_c(t) + D_{AB}(t) \quad (4.23)$$

$$B'(t) = B_c(t)L_c(t) + D_B(t) \tag{4.24}$$

$$\tilde{A}'(t) = A_c(t) - K_c(t)H_c(t) + D_{BM}(t) \tag{4.25}$$

This has not altered the solution to $\Lambda_X(t)$ and $\Lambda_P(t)$, which implies that $V(x(t),e(t),t)$ is still positive definite. However, the sufficient condition for $\dot{V}$ to be negative semidefinite is that the right side of equation (4.22) be negative definite for $x(t)=\emptyset$. i.e.

$$-\begin{bmatrix} L_c^T(t)R(t)L_c(t)+Q_c(t) & -L_c^T(t)R(t)L_c(t) \\ -L_c^T(t)R(t)L_c(t) & L_c^T(t)R(t)L_c(t)+Q_e(t) \end{bmatrix}$$

$$+\begin{bmatrix} \Lambda_X(t)D_{AB}+D_{AB}^T\Lambda_X(t) & \Lambda_X(t)D_B+D^T\Lambda_P(t) \\ \Lambda_P(t)D+D_B^T\Lambda_X(t) & \Lambda_P(t)D_{BM}+D_{BM}^T\Lambda_P(t) \end{bmatrix} < \emptyset \tag{4.26}$$

This inequality constraint becomes the sufficient condition for the closed-loop system to be stable in the sense of Lyapunov. It is possible that for certain ranges of parameter uncertainty, the inequality constraint (equation 4.26) could be violated.

4.2.2 Lyapunov Function Derivation

The continuous, linear, time-varying closed-loop system dynamics with parameter uncertainties are

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} (A_C(t)-B_C(t)L_C(t)+D_{AB}(t)) & (B_C(t)L_C(t)+D_B(t)) \\ D(t) & (A_C(t)-K_C(t)H_C(t)+D_{BM}(t)) \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{4.27}$$

For the linear, time-invariant system, it is easy to see how the uncertainty parameters (D, D_{BM}, D_{AB}, and D_B) can affect the stability of the system. Without these parameters, the stability of the system is characterized by the eigenvalues of the controller and the observer, designed separately [122]. This can be used to determine acceptable bounds of parameter variations, under which the system remains stable. It is desirable to provide the same kind of stability analysis, given parameter variations, for linear, time-varying systems. This is accomplished through the following derivation of a Lyapunov function that accounts for variations in parameters. Using the same optimization technique set up in the previous section, the performance index is chosen as

$$J = x_f^T G_f x_f$$

$$+ \int_{0}^{t_f} [x^T(t) Q_c(t) x(t) + e^T(t) Q_e(t) e(t)$$

$$+ u^T(t) R(t) u(t)]\ dt \qquad (4.28)$$

subject to

$$\dot{x}(t) = \bar{A}'(t) x(t) + B'(t) e(t) + \Gamma(t) \omega(t) \qquad (4.29)$$

$$\dot{e}(t) = D(t) x(t) + \tilde{A}'(t) e(t) - K_c(t) V(t) + \Gamma(t) \omega(t) \qquad (4.30)$$

where

$$\omega(t) \sim N(0, \bar{Q}(t) \delta(t - \tau)) \quad ,$$

$$V(t) \sim N(0, \bar{R}(t) \delta(t - \tau)) \qquad (4.31)$$

$$u(t) = -L_c(t) [x(t) - e(t)] \qquad (4.32)$$

Rewriting the performance index in terms of x(t) and e(t), and carrying through the expectation, equation (4.28) becomes

$$E\{J\} = \mathrm{tr}\begin{bmatrix} G_f & \emptyset \\ \emptyset & \emptyset \end{bmatrix}\begin{bmatrix} X_f & S_f \\ S_f^T & P_f \end{bmatrix}$$

$$+ \int_{\emptyset}^{t_f} \mathrm{tr}\begin{bmatrix} L_c^T(t)R(t)L_c(t)+Q_c(t) & -L_c^T(t)R(t)L_c(t) \\ -L_c^T(t)R(t)L_c(t) & L_c^T(t)R(t)L_c(t)+Q_e(t) \end{bmatrix}\begin{bmatrix} X(t) & S(t) \\ S^T(t) & P(t) \end{bmatrix}dt \qquad (4.33)$$

where

$$X(t) = E[x(t)x^T(t)] \quad , \quad S(t) = E[x(t)e^T(t)] \quad ,$$

$$P(t) = E[e(t)e^T(t)] \qquad (4.34)$$

Using the same technique in Appendix B, the differential equations for $X(t)$, $S(t)$, and $P(t)$ are

$$\begin{bmatrix} \dot{X}(t) & \dot{S}(t) \\ \dot{S}^T(t) & \dot{P}(t) \end{bmatrix} = \begin{bmatrix} \overline{A}'(t) & B'(t) \\ D(t) & \tilde{A}(t) \end{bmatrix}\begin{bmatrix} X(t) & S(t) \\ S^T(t) & P(t) \end{bmatrix}$$

$$+ \begin{bmatrix} X(t) & S(t) \\ S^T(t) & P(t) \end{bmatrix}\begin{bmatrix} \overline{A}'^T(t) & D^T(t) \\ B'^T(t) & \tilde{A}'^T(t) \end{bmatrix}$$

$$+ \begin{bmatrix} \overline{Q}(t) & \overline{Q}(t) \\ \overline{Q}(t) & K(t)\overline{R}(t)K^T(t)+\overline{Q}(t) \end{bmatrix} \qquad (4.35)$$

Following the same steps in Section 3.2.1, cost function can be rewritten as

$$J' = E[V(x,e,t)] + \int_{\emptyset}^{t_f} \mathrm{tr}\Lambda Q d\tau \qquad (4.36)$$

where V is a Lyapunov function of the form

$$V(x(t),e(t),t) = [x^T(t) \;\; e^T(t)] \begin{bmatrix} \Lambda_X(t) & \Lambda_S(t) \\ \Lambda_S^T(t) & \Lambda_P(t) \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \qquad (4.37)$$

where

$$\begin{bmatrix} \dot{\Lambda}_X(t) & \dot{\Lambda}_S(t) \\ \dot{\Lambda}_S^T(t) & \dot{\Lambda}_P(t) \end{bmatrix} = -\begin{bmatrix} \Lambda_X(t) & \Lambda_S(t) \\ \Lambda_S^T(t) & \Lambda_P(t) \end{bmatrix} \begin{bmatrix} \bar{A}'(t) & B'(t) \\ D(t) & \tilde{A}'(t) \end{bmatrix}$$

$$- \begin{bmatrix} \bar{A}'^T(t) & D^T(t) \\ B'^T(t) & \tilde{A}'^T(t) \end{bmatrix} \begin{bmatrix} \Lambda_X(t) & \Lambda_S(t) \\ \Lambda_S^T(t) & \Lambda_P(t) \end{bmatrix}$$

$$- \begin{bmatrix} L_c^T(t)R(t)L_c(t)+Q_c(t) & -L_c^T(t)R(t)L_c(t) \\ -L_c^T(t)R(t)L_c(t) & L_c^T(t)R(t)L_c(t)+Q_e(t) \end{bmatrix} ,$$

$$\begin{bmatrix} \Lambda_X(t_f) & \Lambda_S(t_f) \\ \Lambda_S^T(t_f) & \Lambda_P(t_f) \end{bmatrix} = \begin{bmatrix} G_f & \emptyset \\ \emptyset & \emptyset \end{bmatrix} \qquad (4.38)$$

With parameter uncertainties in the Lyapunov function, $\Lambda_X(t)$ is no longer equivalent to the controller Riccati matrix which implies

$$L_c(t) \neq R^{-1}(t)B_c^T(t)\Lambda_x(t) \qquad (4.39)$$

In addition, $\Lambda_S(t)$ is not equal to zero even with $\Lambda_S(t_f)$ equal to zero. Therefore, the Lyapunov function (equation 4.38) cannot be simplified.

4.2.3 Lyapunov Function Validation

Proposition 4.1

V(x,e,t) from equation (4.37) satisfies the four sufficiency conditions for asymptotic stability in the sense of Lyapunov for the continuous, time-varying system described in equations (2.64)-(2.66) when subjected to system parameter uncertainties of the form described in equations (4.14), (4.15), (4.18), and (4.19) under the assumptions that $(A(t),B(t))$ and $(A(t),\Gamma(t))$ are controllable and $(A(t),H(t))$ and $(A(t),Q_c^{1/2}(t))$ are observable [121]. Thus, V(x,e,t) is positive definite and bounded, i.e.,

$$0 < V(x(t),e(t),t) \leq \beta(||x||,||e||) \qquad (4.40)$$

where

$$V(x(t),e(t),t) = [x^T(t) \quad e^T(t)]\left[\Phi(t,t_f)\bar{G}_f\Phi^T(t,t_f)\right.$$

$$\left. + \int_t^{t_\emptyset} \Phi(t,\tau)\tilde{Q}(\tau)\Phi^T(t,\tau)d\tau\right]\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} ,$$

$$\bar{G}_f = \begin{bmatrix} G_f & \emptyset \\ \emptyset & \emptyset \end{bmatrix} ,$$

$$\tilde{Q}(t) = \begin{bmatrix} L_c^T(t)R(t)L_c(t)+Q_c(t) & -L_c^T(t)R(t)L_c(t) \\ -L_c^T(t)R(t)L_c(t) & L_c^T(t)R(t)L_c(t)+Q_e(t) \end{bmatrix} \quad (4.41)$$

and $\Phi(t_\emptyset,\tau)$ represents the state transition matrix for the system defined in equation (4.27) and β is a continuous nondecreasing scalar valued function.

Proof:

For $R(t)>\emptyset$, $Q_c(t)\geq\emptyset$, and $Q_e(t)\geq\emptyset$, then $\tilde{Q}(t)\geq\emptyset$ as shown in equation (3.39). This assures that equation (4.41) is nonnegative definite for $t\geq t_\emptyset$ [25]. In order that $V(x,e,t)$ satisfies the sufficiency conditions for asymptotic stability in the sense of Lyapunov, the inequality constraint (equation 4.40) must be satisfied. With equation (4.40), the first three requirements for the Lyapunov function are satisfied. Taking the derivative of equation (4.37)

$$\dot{V}(x(t),e(t),t) = [x^T(t)\ \ e^T(t)]\begin{bmatrix} -L_c^T(t)R(t)L_c(t)-Q_c(t) & L_c^T(t)R(t)L_c(t) \\ L_c^T(t)R(t)L_c(t) & -L_c^T(t)R(t)L_c(t)-Q_e(t) \end{bmatrix}\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \qquad (4.42)$$

This is similiar to equation (3.38) and is negative definite for $x(t)\neq 0$ and given any uncertainty of the form described in equations (4.14), (4.15), (4.18), and (4.19). For equation (4.37) to be a valid Lyapunov function for all x(t) and e(t), equation (4.40) must be satisfied.

4.3 Discrete-Time Linear Systems

4.3.1 Extension of Lyapunov Function From Section 3.3.1

The Lyapunov function for the discrete-time, linear system without parameter uncertainties is rewritten here as

$$V_K = [x_K^T\ e_K^T]\begin{bmatrix} \Lambda_{X_K} & 0 \\ 0 & \Lambda_{P_K} \end{bmatrix}\begin{bmatrix} x_K \\ e_K \end{bmatrix} \qquad (4.43)$$

where

$$\Lambda_{X_K} = \bar{A}_K^T \Lambda_{X_{K+1}} \bar{A}_K + L_K^T R_K L_K + Q_{c_K}\ , \quad \Lambda_{X_N} = G_N \qquad (4.44)$$

$$\Lambda_{P_K} = \tilde{A}_K^T \Lambda_{P_{K+1}} \tilde{A}_K + L_K^T (B_K^T \Lambda_{X_{K+1}} B_K + R_K) L_K$$

$$+ Q_{e_K} \ , \ \Lambda_{P_N} = 0 \tag{4.45}$$

The discrete-time, linear closed-loop system dynamics are

$$x_{K+1} = (A_K - B_K L_{c_K}) x_K + B_K L_{c_K} e_K \tag{4.46}$$

$$\hat{x}_{K+1} = (A_{c_K} - B_{c_K} L_{c_K}) \hat{x}_K$$

$$+ K_{c_{K+1}} [y_K - H_{c_{K+1}} \bar{x}_{K+1} + M_{c_{K+1}} L_{c_{K+1}} \hat{x}_{K+1}] \tag{4.47}$$

$$y_K = H_K x_K - M_K L_{c_K} \hat{x}_K \tag{4.48}$$

$$\bar{x}_K = (A_{c_K} - B_{c_K} L_{c_K}) \hat{x}_K \tag{4.49}$$

$$K_{c_K} = \tilde{P}_{c_K} H_{c_K}^T [H_{c_K} \tilde{P}_{c_K} H_{c_K}^T + R_K]^{-1} \tag{4.50}$$

where A_K, B_K, H_K, and M_K are the unknown true system parameters, and A_{c_K}, B_{c_K}, H_{c_K}, M_{c_K}, K_{c_K}, and L_{c_K} are the designed (or nominal) system parameters. The discrete modelling errors (or deviations from the nominal) are defined as

$$\Delta A_K = A_K - A_{c_K} \qquad (4.51)$$

$$\Delta B_K = B_K - B_{c_K} \qquad (4.52)$$

$$\Delta H_K = H_K - H_{c_K} \qquad (4.53)$$

$$\Delta M_K = M_K - M_{c_K} \qquad (4.54)$$

The system dynamic can be rewritten to emphasize the modelling error in the following way

$$x_{K+1} = (A_{c_K} - B_{c_K} L_{c_K} + D_{AB_K}) x_K + (B_{c_K} L_{c_K} + D_{B_K}) e_K \qquad (4.55)$$

where

$$D_{AB_K} = (A_K - A_{c_K}) - (B_K - B_{c_K}) L_{c_K} \qquad (4.56)$$

$$D_{B_K} = (B_K - B_{c_K}) L_{c_K} \qquad (4.57)$$

The discrete estimation error, defined as

$$e_K = x_F - \hat{x}_K \qquad (4.58)$$

has the following dynamics

$$e_K = D_K x_K + (A_{c_K} - K_{c_{K+1}} H_{c_{K+1}} A_{c_K} + D_{BM_K}) e_K \qquad (4.59)$$

where

$$D_K = (A_K - A_{c_K}) - (B_K - B_{c_K})L_{c_K}$$

$$- K_{c_{K+1}}(H_{K+1} - H_{c_{K+1}})A_K + K_{c_{K+1}}(M_{K+1} - M_{c_{K+1}})L_{c_{K+1}}A_K \quad (4.60)$$

$$D_{BM_K} = (B_K - B_{c_K})L_{c_K} - K_{c_{K+1}}(M_{K+1} - M_{c_{K+1}})L_{c_{K+1}}A_K \quad (4.61)$$

Therefore, the discrete closed-loop system is

$$\begin{bmatrix} x_{K+1} \\ e_{K+1} \end{bmatrix} =$$

$$\begin{bmatrix} (A_{c_K} - B_{c_K}L_{c_K} + D_{AB_K}) & (B_{c_K}L_{c_K} + D_{B_K}) \\ D_K & (A_{c_K} - K_{c_{K+1}}H_{c_{K+1}}A_{c_K} + D_{BM_K}) \end{bmatrix} \begin{bmatrix} x_K \\ e_K \end{bmatrix} \quad (4.62)$$

Equation (4.44) and (4.45) can be rewritten as

$$\begin{bmatrix} \bar{A}_K^T & 0 \\ L_K^T B_K^T & \tilde{A}_K^T \end{bmatrix} \begin{bmatrix} \hat{X}_{K+1} & 0 \\ 0 & \hat{P}_{K+1} \end{bmatrix} \begin{bmatrix} \bar{A}_K & B_K L_K \\ 0 & \tilde{A}_K \end{bmatrix}$$

$$- \begin{bmatrix} \hat{X}_K & 0 \\ 0 & \hat{P}_K \end{bmatrix} = - \begin{bmatrix} L_K^T R_K L_K + Q_{c_K} & -L_K^T R_K L_K \\ -L_K^T R_K L_K & L_K^T R_K L_K + Q_{e_K} \end{bmatrix} \quad (4.63)$$

Incorporating the effects of the modelling errors into equation (4.63), the result is

$$
\begin{bmatrix} \bar{A}_K{}'^{T} & D_K^T \\ B_K{}'^{T} & \tilde{A}_K{}'^{T} \end{bmatrix}
\begin{bmatrix} \Lambda_{X_{K+1}} & 0 \\ 0 & \Lambda_{P_{K+1}} \end{bmatrix}
\begin{bmatrix} \bar{A}_K{}' & B_K{}' \\ D_K & \tilde{A}_K{}' \end{bmatrix}
$$

$$
- \begin{bmatrix} \Lambda_{X_K} & 0 \\ 0 & \Lambda_{P_K} \end{bmatrix} = - \begin{bmatrix} L_{c_K}^T R_K L_{c_K} + Q_{c_K} & -L_{c_K}^T R_K L_{c_K} \\ -L_{c_K}^T R_K L_{c_K} & L_{c_K}^T R_K L_{c_K} + Q_{e_K} \end{bmatrix}
$$

$$
+ \begin{bmatrix} D_{AB_K}^T & D_{B_K}^T \\ D_K^T & D_{BM_K}^T \end{bmatrix}
\begin{bmatrix} \Lambda_{X_{K+1}} & 0 \\ 0 & \Lambda_{P_{K+1}} \end{bmatrix}
\begin{bmatrix} \bar{A}_K{}' & B_K{}' \\ D_K & \tilde{A}_K{}' \end{bmatrix}
$$

$$
+ \begin{bmatrix} \bar{A}_K{}'^{T} & D_K^T \\ B_K{}'^{T} & \tilde{A}_K{}'^{T} \end{bmatrix}
\begin{bmatrix} \Lambda_{X_{K+1}} & 0 \\ 0 & \Lambda_{P_{K+1}} \end{bmatrix}
\begin{bmatrix} D_{AB_K} & D_{B_K} \\ D_K & D_{BM_K} \end{bmatrix} \qquad (4.64)
$$

As in the continuous-time case, this does not alter the solution to Λ_{X_K} and Λ_{P_K}, which implies that V_K is still positive definite for any parameter variations. However, the sufficient condition for ΔV_K to be negative semidefinite is

$$
-\begin{bmatrix} L_{c_K}^T R_K L_{c_K} + Q_{c_K} & -L_{c_K}^T R_K L_{c_K} \\ -L_{c_K}^T R_K L_{c_K} & L_{c_K}^T R_K L_{c_K} + Q_{e_K} \end{bmatrix}
$$

$$
+\begin{bmatrix} D_{AB_K}^T & D_K^T \\ D_{B_K}^T & D_{BM_K}^T \end{bmatrix}\begin{bmatrix} \hat{X}_{K+1} & 0 \\ 0 & \hat{P}_{K+1} \end{bmatrix}\begin{bmatrix} \bar{A}_K' & B_K' \\ D_K & \tilde{A}_K' \end{bmatrix}
$$

$$
+\begin{bmatrix} \bar{A}_K'^T & D_K^T \\ B_K'^T & \tilde{A}_K'^T \end{bmatrix}\begin{bmatrix} \hat{X}_{K+1} & 0 \\ 0 & \hat{P}_{K+1} \end{bmatrix}\begin{bmatrix} D_{AB_K} & D_{B_K} \\ D_K & D_{BM_K} \end{bmatrix} \leq 0 \quad (4.65)
$$

This inequality constraint becomes the sufficient condition for the discrete closed-loop system to be stable in the sense of Lyapunov. For certain ranges of parameter uncertainties, it is possible that this inequality constraint could be violated.

4.3.2 Lyapunov Function Derivation

The discrete-time, linear closed-loop system dynamics with parameter uncertainties are

$$\begin{bmatrix} x_{K+1} \\ e_{K+1} \end{bmatrix} = \begin{bmatrix} (A_{c_K} - B_{c_K} L_{c_K} + D_{AB_K}) & (B_{c_K} L_{c_K} + D_{B_K}) \\ D_K & (A_{c_K} - K_{c_{K+1}} H_{c_{K+1}} A_{c_K} + D_{BM_K}) \end{bmatrix} \begin{bmatrix} x_K \\ e_K \end{bmatrix} \tag{4.66}$$

A means of analyzing the stability of linear, discrete-time systems subject to parameter variations in developed in this section. This is accomplished through the following derivation of a Lyapunov function which accounts for variations in parameters. The discrete LQG performance index is chosen as

$$J = x_N^T G_N x_N + \sum_{K=0}^{N-1} x_K^T Q_{c_K} x_K + e_K^T Q_{e_K} e_K + u_K^T R_K u_K \tag{4.67}$$

subject to

$$x_{K+1} = \bar{A}_K' x_K + B_K' e_K + \Gamma_K \omega_K \tag{4.68}$$

$$e_{K+1} = D_K x_K + \tilde{A}_K' e_K - K_{K+1} \nu_{K+1} + \Gamma_K \omega_K \tag{4.69}$$

where

$$\omega_K \sim N(\emptyset, \bar{Q}_K) \quad , \quad \nu_K \sim N(\emptyset, \bar{R}_K) \tag{4.70}$$

$$\bar{A}_K' = A_{c_K} - B_{c_K} L_{c_K} + D_{AB_K} \tag{4.71}$$

$$B_K' = B_{c_K} L_{c_K} + D_{B_K} \tag{4.72}$$

$$\tilde{A}_K' = A_{c_K} - K_{c_{K+1}} H_{c_{K+1}} A_{c_K} + D_{BM_K} \tag{4.73}$$

$$u_K = -L_{c_K}(x_K - e_K) \tag{4.74}$$

Rewriting the discrete performance index in terms of x_K and e_K, and carrying through the expectation gives

$$E\{J\} = \mathrm{tr}\begin{bmatrix} G_N & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X_N & S_N \\ S_N^T & P_N \end{bmatrix}$$

$$+ \sum_{K=0}^{N-1} \mathrm{tr}\begin{bmatrix} L_{c_K}^T R_K L_{c_K} + Q_{c_K} & -L_{c_K}^T R_K L_{c_K} \\ -L_{c_K}^T R_K L_{c_K} & L_{c_K}^T R_K L_{c_K} + Q_{e_K} \end{bmatrix}\begin{bmatrix} X_K & S_K \\ S_K^T & P_K \end{bmatrix} \tag{4.75}$$

where

$$X_K = E[x_K x_K^T] \quad , \quad S_K = E[x_K e_K^T] \quad , \quad P_K = E[e_K e_K^T] \tag{4.76}$$

Using the same technique in Appendix B, the difference equations for X_K, S_K, and P_K are

$$\begin{bmatrix} X_{K+1} & S_{K+1} \\ S^T_{K+1} & P_{K+1} \end{bmatrix} = \begin{bmatrix} \bar{A}_K' & B_K' \\ D_K & \tilde{A}_K' \end{bmatrix} \begin{bmatrix} X_K & S_K \\ S^T_K & P_K \end{bmatrix} \begin{bmatrix} \bar{A}_K'^T & D^T_K \\ B_K'^T & \tilde{A}_K'^T \end{bmatrix}$$

$$+ \begin{bmatrix} \bar{Q}_K & \bar{Q}_K \\ \bar{Q}_K & K_{c_{K+1}} \bar{R}_{K+1} K^T_{c_{K+1}} \end{bmatrix} \qquad (4.77)$$

Following the same steps in Section 3.3.2, the cost function can be rewritten as

$$J' = E[V_K] + \sum_{i=1}^{N-1} tr\Lambda_i Q_i \qquad (4.78)$$

where V_K is the discrete Lyapunov function of the form

$$V_K(x_K, e_K, t_K) = [x^T_K \; e^T_K] \begin{bmatrix} \Lambda x_K & \Lambda s_K \\ \Lambda s^T_K & \Lambda p_K \end{bmatrix} \begin{bmatrix} x_K \\ e_K \end{bmatrix} \qquad (4.79)$$

where

$$\begin{bmatrix} \Lambda_{X_K} & \Lambda_{X_K} \\ \Lambda_{S}^{T_K} & \Lambda_{P_K} \end{bmatrix} = \begin{bmatrix} \bar{A}_K^{',T} & D_K^T \\ B_K^{',T} & \tilde{A}_K^{',T} \end{bmatrix} \begin{bmatrix} \Lambda_{X_{K+1}} & \Lambda_{X_{K+1}} \\ \Lambda_{S_{K+1}}^T & \Lambda_{P_{K+1}} \end{bmatrix} \begin{bmatrix} \bar{A}_K' & B_K' \\ D_K & \tilde{A}_K' \end{bmatrix}$$

$$+ \begin{bmatrix} L_{c_K}^T R_K L_{c_K} + Q_{c_K} & -L_{c_K}^T R_K L_{c_K} \\ -L_{c_K}^T R_K L_{c_K} & L_{c_K}^T R_K L_{c_K} + Q_{e_K} \end{bmatrix} ,$$

$$\begin{bmatrix} \Lambda_{X_N} & \Lambda_{S_N} \\ \Lambda_{S_N}^T & \Lambda_{P_N} \end{bmatrix} = \begin{bmatrix} G_N & 0 \\ 0 & 0 \end{bmatrix} \tag{4.80}$$

As noted in Section 4.2.2, Λ_{X_K} is no longer equivalent to the controller Riccati matrix with parameter uncertainties present which implies

$$L_{c_K} \neq (R_K + B_{c_K}^T \Lambda_{X_{K+1}} B_{c_K})^{-1} B_{c_K}^T \Lambda_{X_{K+1}} A_{c_K} \tag{4.81}$$

In addition, $\Lambda_{S_K} \neq 0$ for $K = N-1,\ldots,0$, even with $\Lambda_{S_N} = 0$. Therefore the Lyapunov function cannot be simplified.

4.3.3 Lyapunov Function Validation

Proposition 4.2

$V_K(x_K,e_K,t_K)$ from equation (4.79) satisfies the sufficiency conditions for asymptotic stability in the sense of Lyapunov for the discrete, time-varying system described in equations (2.75)-(2.76) when subjected to system parameter uncertainties of the form described in equations (4.56), (4.57), (4.60), and (4.61) under the assumptions that (A_K, B_K) and (A_K, Γ_K) are controllable and (A_K, H_K) and $(A_K, Q_{c_K}^{1/2})$ are observable. Thus, $V_K(x_K, e_K, t_K)$ is positive definite and bounded, i.e.,

$$0 < V_K = [x_K^T \;\; e_K^T] \begin{bmatrix} \Lambda_{x_K} & \Lambda_{S_K} \\ \Lambda_{S_K}^T & \Lambda_{P_K} \end{bmatrix} \begin{bmatrix} x_K \\ e_K \end{bmatrix} \leq \beta_K(||x_K||, ||e_K||) \qquad (4.82)$$

and $\Delta V_K \neq 0$ except at $x_K = 0$ [96], where β_K is a nondecreasing scalar valued function.

Proof:

For $R_K > 0$, $Q_{c_K} \geq 0$, and $Q_{e_K} \geq 0$, then $\tilde{Q} \geq 0$, where

$$\tilde{Q}_K = \begin{bmatrix} L_{c_K}^T R_K L_{c_K} + Q_{c_K} & -L_{c_K}^T R_K L_{c_K} \\ -L_{c_K}^T R_K L_{c_K} & L_{c_K}^T R_K L_{c_K} + Q_{e_K} \end{bmatrix} \qquad (4.83)$$

For $\tilde{Q}_K \geq 0$ and $G_N \geq 0$, then $\Lambda_K \geq 0$ for $K \geq 0$. For equation (4.79) to satisfy the sufficiency conditions for asymptotic stability in the sense of Lyapunov, Λ_K must be positive definite for $K > 0$ and all nonzero x_K and e_K and bounded from above. The first three requirements for a

Lyapunov function are satisfied if equation (4.82) is valid. The difference equation of V_K is

$$\Delta V_K = V_{K+1} - V_K$$

$$= [x_K^T \; e_K^T] \begin{bmatrix} -L_{c_K}^T R_K L_{c_K} - Q_{c_K} & L_{c_K}^T R_K L_{c_K} \\ L_{c_K}^T R_K L_{c_K} & -L_{c_K}^T R_K L_{c_K} - Q_{e_K} \end{bmatrix} \begin{bmatrix} x_K \\ e_K \end{bmatrix} \qquad (4.84)$$

This is similiar to equation (3.76), and is negative definite for $x_K \neq 0$ and for any uncertainty of the form described in equations (4.56), (4.57), (4.60), and (4.61), since equation (3.76) is independant of the variations. For equation (4.79) to be a valid Lyapunov function for all x_K and e_K, equation (4.82) must be satisfied.

4.4 Practicality of This Derived Lyapunov Function

Section 4.2.3 and 4.3.3 have provided sufficiency conditions under which equations (4.37) and (4.79) are valid Lyapunov functions. Considering the discrete-time case only, equation (4.80) is a backward difference equation of the form

$$P_K = \hat{A}_K^T P_{K+1} \hat{A}_K + Q_K \qquad (4.85)$$

$$\Lambda_N = G_N \tag{4.86}$$

where $K=N-1,\ldots,\emptyset$. $\hat{A}_K$ represents a state transition matrix. In addition, Q_K is positive semidefinite from equation (3.78). Stability in the sense of Lyapunov is only applicable to an infinite time problem. For the finite-time problem, the Lyapunov function does not provide a measure of stability; however, by investigating the time response of the Lyapunov equation, a measure of system performance can be obtained. In particular, the Lyapunov equation becomes unbounded from above when the system performs poorly.

With $G_N \geq \emptyset$ and $\hat{Q}_K \geq \emptyset$, Λ_K will always be at least positive semidefinite. Variations in the systems parameters will not cause V_K to become nonpositive definite, nor will it cause ΔV_K to become nonnegative semidefinite. However, these variations can cause V_K to become unbounded from above.

When the variations in system parameters become large enough to cause the system to diverge (or perform poorly), the solution to the Lyapunov equation goes to infinity (becomes unbounded from above). This characteristic of the Lyapunov equation is useful in providing a measure of system performance for the linear, time-varying, finite-time problem.

For the linear, time-invariant problem, it is possible to analyze the steady-state value of the Lyapunov function. In steady-state, equation (4.85) becomes

$$\Lambda = \hat{A}_{SS}^{T}\Lambda\hat{A}_{SS} + \hat{Q}_{SS} \qquad (4.87)$$

where $()_{SS}$ represents steady-state values. It is now possible that, for certain regions of parameter variations, Λ may not be positive definite.

SECTION V

MISSILE OBSERVER PERFORMANCE IMPROVEMENTS THROUGH OPTIMAL FEEDBACK CONTROL

5.1 Introduction

The application of Linear Quadratic Gaussian (LQG) optimal control theory to the tactical missile guidance problem has drawn much attention in recent years. It has been demonstrated that for short range tactical missiles, the LQG guidance law provides significant performance improvements over the more commonly used classical proportional navigation (pro-nav) guidance laws [109].

A critical issue that affects the performance of the LQG guidance law is the fact that it is a function of missile-to-target position, velocity, and acceleration, and time-to-go. Time-to-go is usually approximated as a function of the position, velocity, and acceleration. A more detailed discussion is presented in Section VI. In the derivation of the guidance law it is assumed that this information is accurate and available on board the missile. Most present day missiles can obtain a measure of the missile's acceleration through on-board accelerometers. In addition, passive seekers are used to provide a measure of line-of-sight angle and rate.

Extended Kalman filters have been used to estimate the needed guidance information from the information available on board the missile with very good results in terms of minimizing miss distance at final time [109]. However, in many instances, the estimates from the filters have not been very good, partially because it is impossible to accurately model the target acceleration. Although not the subject of this dissertation, much work has been accomplished toward improving target acceleration modelling.

In addition, certain missile/target engagements reduce the observability of the filter states; thus, degrading the performance of the filter, and in turn, the guidance law. The emphasis in this section is to improve the state estimates through the guidance law. As in the previous sections, an observer will be used instead of a filter algorithm.

The task is to incorporate an additional term in the LQG performance index, which is developed to minimize final miss distance while minimizing control effort. This new term is included to maximize the observability Grammian matrix of the observer, i.e.,the measure of the uncertainty of the state estimates. This new term will require the guidance law to minimize the error variance matrix of the observer. This is similar to the efforts by Hull, Speyer, Tseng, and Larson [63,123], in which they developed a guidance law using the LQG performance index which included a term that would maximize the information matrix. This guidance law could not be solved in closed form requiring the use of a numerical optimization program. The results, however, did show that the guidance law could improve the filter algorithm's performance while attempting to hit the target.

The impetus for this work comes from the Lyapunov stability analysis of the pseudomeasurement observer (PMO) in Section II. By taking advantage of the PMO's algorithm, a closed form solution is obtainable.

5.2 Missile Model

The state dynamics model used for the development of both the missile's guidance law is linear and the estimation algorithm is nonlinear and they are set up in rectangular coordinates as follows:

$$\dot{x} = Ax + Bu \tag{5.1}$$

where

$$A = \begin{bmatrix} \emptyset & I & \emptyset \\ \emptyset & 0 & I \\ \emptyset & \emptyset & -\lambda_T \end{bmatrix} \tag{5.2}$$

$$B = \begin{bmatrix} \emptyset \\ -I \\ \emptyset \end{bmatrix} \tag{5.3}$$

and x consists of the three components of missile-to-target position, velocity, and target acceleration in inertial coordinates.

The line-of-sight angles, measured from a passive seeker, are azimuth, (Θ), and elevation, (ϕ), angles. The relationship between these angles and the observer's states is illustrated in Figure 5.1.

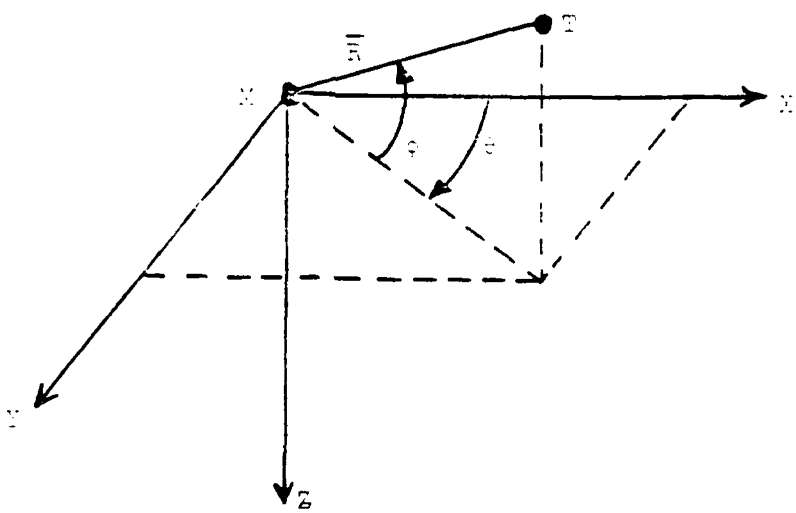


Figure 5.1, Angular Measurements Related to Observer States

The nonlinear functions relating the angles to the states in a rectangular coordinate frame are

$$\phi = \tan^{-1}\left[\frac{-Z}{\sqrt{X^2 + Y^2}}\right] \tag{5.4}$$

$$\theta = \tan^{-1}\left[\frac{Y}{X}\right] \tag{5.5}$$

where X, Y, and Z are the three components of relative position in inertial coordinates.

For the PMO, the measurement model is rewritten as [119,120]

$$y = H(z)x \tag{5.6}$$

where

$$H(z) = \begin{bmatrix} \sin\theta & -\cos\theta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\theta & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.7}$$

5.3 Optimization Problem

Consider the following performance index

$$J = \frac{1}{2}\int_0^{t_f} (u^T R u - x^T Q x)\, dt \tag{5.8}$$

subject to

$$D x_f = 0 \tag{5.9}$$

$$\dot{x} = Ax + Bu \quad , \quad x(0) = x_0 \tag{5.10}$$

where t_f is given, R is positive definite, and Q is positive semidefinite. This performance criteria is chosen to require the control law to drive the system to a zero terminal miss while minimizing the control effort. In addition, there is the term (x^TQx) which is maximized over time. This term is constructed to maximize some measure of the observability Grammian matrix of the observer; thus, minimizing the error variance matrix of the observer. The differential equation for the observability Grammian matrix for the PMO is

$$\dot{P} - AP - PA^T + PH^T(z)V^{-1}H(z)P - W = 0 \qquad (5.11)$$

where V is the power spectral density of the measurements and W is the power spectral density of the observer states. Taking the inverse of the observability Grammian matrix, the differential equation becomes

$$\dot{P}^{-1} + P^{-1}A + A^TP^{-1} - H^T(z)V^{-1}H + P^{-1}WP^{-1} = 0 \qquad (5.12)$$

The results of the Lyapunov stability analysis of Section II showed that by decreasing V (or increasing V^{-1}) , the inverse observability Grammian matrix (P^{-1}) would increase. Therefore, the performance index should include a term to maximize $H^T(z)V^{-1}H(z)$, where H(z) is defined in equation (5.7). The measurement power spectral density is assumed to be

$$V = \alpha \qquad (5.13)$$

where and α is some positive constant representative of the accuracy of the infrared passive seeker.

Define the second term in equation (5.8) as

$$x^T Q x = \mathrm{tr}\ \{R^2 H^T(x) V^{-1} H(x)\} \qquad (5.14)$$

where R is range and H(X) comes from substituting the following identities from Figure (5.1) into equation (5.7)

$$\sin\theta = \frac{Y}{\sqrt{X^2 + Y^2}} \qquad (5.15)$$

$$\cos\theta = \frac{X}{\sqrt{X^2 + Y^2}} \qquad (5.16)$$

$$\sin\phi = -\frac{Z}{R} \qquad (5.17)$$

$$\cos\phi = \frac{\sqrt{X^2 + Y^2}}{R} \qquad (5.18)$$

Note that

$$\mathrm{tr}\ \{R^2 H^T(x) V^{-1} H(x)\} = \mathrm{tr}\ \{R^2 V^{-1} H(x) H^T(x)\}$$

$$= \mathrm{tr}\ \frac{R^2}{\alpha}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{tr}\ (x^T \frac{2}{\alpha} x)$$

$$= x^T \frac{2}{\alpha} x \tag{5.19}$$

With equation (5.14), the following definition of Q can be made

$$Q = \begin{bmatrix} 2\alpha^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.20}$$

The solution to this optimization problem is [38]

$$u = -R^{-1} B^T (S - G Q^{-1} G^T) x \tag{5.21}$$

where

$$\dot{S} = -SA - A^T S + SBR^{-1} B^T S + Q\ ,\ S(t_f) = 0 \tag{5.22}$$

$$\dot{G} = -(A^T - SBR^{-1} B^T) G\ ,\ G(t_f) = D^T \tag{5.23}$$

$$\dot{Q} = G^T B R^{-1} B^T G\ ,\ Q(t_f) = 0 \tag{5.24}$$

5.4 Design Considerations

To ensure that no conjugate points exist,

$(S - GQ^{-1}G^T)$ must be finite for $t_\emptyset \leq t < t_f$. This also satisfies the same Riccati equation as S (equation 5.22). With Q positive semidefinite, it is possible for S to blow up if integrated over a long period of time. This may or may not cause $(S - GQ^{-1}G^T)$ to blow up [26]. This potential problem puts some restrictions on the final time boundary condition, t_f. $\dot{S}$ is a backward Riccati differential equation and it is important that the critical time, t_c (where the conjugate point occurs), does not fall between the integration period $[t_f, t_\emptyset)$.

In the homing missile problem, initial time, $t_\emptyset$, is known but final time, t_f, is not known. A restriction on t_f such that no conjugate point occurs is

$$t_f - t_c > t_f - t_\emptyset = t_g \tag{5.25}$$

or

$$t_f > t_c + t_g \tag{5.26}$$

where t_g is time-to-go. The conjugate point is avoided through the selection of the guidance parameter, α. A smaller value of α will lesson the rate of change of S, such that a larger interval of $[t_f, t_\emptyset)$ will not contain a conjugate point. A smaller value of α implies the measurement device is more accurate. It also reduces the emphasis for the guidance law to improve the observer's performance. This is a design parameter which may differ from missile system to missile system.

SECTION VI

APPLICATIONS

6.1 Introduction

The purpose of this section is to demonstrate the usefulness of the Lyapunov functions derived in Sections II, III, and IV to measure the stability characteristics (or performance for time-varying systems) of various closed-loop systems given state modelling errors. Several simple problems are used to obtain insight as to how useful the Lyapunov functions are. The analysis is broken up into the following classes of problems: Linear, Time-Invariant Scalar Problem; Linear, Time-Invariant Multivariable Control Problem; Linear, Time-Varying Guidance Problem; and the Homing Missile Guidance Problem with Angle-Only Measurements. In addition, the performance of the LQG guidance law developed to improve the observer's state estimation process as well as minimize miss distance is analyzed.

The linear, time-invariant, scalar problem selected comes from a study by Speyer [122], in which he was able to identify acceptable ranges for state modelling errors (eqns. 4.9-4.12) where in the closed-loop system would remain stable. He accomplished this by rewriting the system equations to emphasize the modelling errors (Section 4.2.1). The acceptable ranges were identified using eigenvalue analysis under steady-state conditions. Since eigenvalue analysis represents both a necessary and sufficient condition for the stability of a linear, time-invariant system, a comparison provides the basis for determining how accurate the Lyapunov functions are at determining stability.

The linear, time-invariant multivariable control problem comes from the work by Doyle and Stein [41], where the closed-loop system is marginally robust. By changing the power spectral density of the state equations for the estimation algorithm, they were able to improve the robustness characteristics of the closed-loop system. This would allow the system to remain stable for larger ranges of the state equation modelling errors. As in the scalar problem, an eigenvalue analysis is performed for the various power spectral densities that Doyle and Stein selected to provide a basis for the Lyapunov function analysis under steady-state conditions. This analysis is performed for a range of state equation modelling errors. In addition, simulation results are obtained to demonstrate the time-varying traits of the closed-loop system.

The impetus for the linear, time-varying guidance work comes from the homing missile guidance problem, which can use the LQG guidance algorithm. The LQG guidance law is a function of missile-to-target position, velocity, and target acceleration, as well as time-to-go. Even for missile systems with the most advanced measuring devices, this information is not readily available, and must be estimated. To do this effectively, the estimation scheme must have an accurate model of the missile/target dynamics. The most difficult information to model is time-to-intercept (or time-to-go) and the target's acceleration. The purpose of this effort is to determine if Lyapunov functions can be used to determine acceptable ranges of time-to-go and target acceleration modelling errors under which the closed-loop homing missile guidance system performs well. Since this is a time-varying problem, it is not possible to look at steady-state conditions; and therefore, eigenvalue analysis cannot be used as a basis for validity. The estimation algorithm used for this study is a linear Kalman observer.

The next example is the homing missile guidance problem with a passive (angle only measuring) seeker. These types of seekers are common for tactical air-to-air and air-to-surface missiles. The LQG guidance algorithm presents a difficult problem for the estimation algorithm, which is needed to estimate missile-to-target position, velocity, and acceleration. For missile systems with passive (angle only) seekers on board, the estimation algorithm has not been very successful in accurately estimating the state information [127]; although, the guidance law has still been successful. The guidance law could be much more successful if the state information were more accurately known. The purpose of the homing missile guidance effort is to determine if Lyapunov functions can be used to identify acceptable ranges of target acceleration modelling errors under which the system performs well.

The difference between this analysis and the linear time-varying guidance problem is that the angle only measurements are nonlinear functions of the system states, and therefore, the estimation algorithm is nonlinear. The estimation algorithm selected for this part of the study is the pseudomeasurement observer (PMO). This algorithm was selected because it exhibits global convergent characteristics [119,120] unlike the more typically used extended Kalman observer (EKO).

The last applications problem is to evaluate the performance of the LQG guidance algorithm developed to improve the estimation algorithm's ability to estimate the state information, as well as minimize the final miss distance (hit the target). The guidance law is designed under the assumption that the missile-to-target position, velocity, and acceleration are available and known perfectly. With the exception of the missile's acceleration, this information is not available on-board a homing missile with angle-only measurements. The guidance law developed in Section V is demonstrated in a missile/target two-degree-of-freedom simulation using the PMO estimation algorithm. The engagement selected for evaluation is the same as that done by Hull, Speyer, Tseng, and Larson [63] so that a comparison can be made. The performance of this system is compared to that of the standard linear quadratic Gaussian (LQG) guidance law by using the Lyapunov function from Section III.

The parameter uncertainty analysis is conducted in Section 6.2. The analysis is conducted for the linear, time-invariant scalar problem (Section 6.2.1), the linear, time-invariant multivariable control problem (Section 6.2.2), the linear, time-varying guidance problem (Section 6.3), the homing missile guidance problem with angle only measurements (Section 6.4), and the homing missile observer performance improvements through the LQG guidance algorithm (Section 6.4.3).

6.2 Parameter Uncertainty Analysis

6.2.1 Linear, Time-Invariant Scalar Problem

The linear, time-invariant problem is as follows [122]

$$\dot{x} = ax + bu + \nu \tag{6.1}$$

$$y = hx + mu + \omega \tag{6.2}$$

$$E[\nu(t)\nu(\tau)] = q_c\delta(t - \tau) \tag{6.3}$$

$$E[\omega(t)\omega(\tau)] = r_c\delta(t - \tau) \tag{6.4}$$

$$E[\nu(t)\omega(\tau)] = 0 \tag{6.5}$$

$$u = -l_c(t)\hat{x} \tag{6.6}$$

$$\dot{\hat{x}} = (a_c - b_c l_c)\hat{x} + k_c(t)[y - h_c\hat{x} + m_c l_c\hat{x}] \tag{6.7}$$

$$l_c(t) = \frac{b_c p_c(t)}{r_c} \tag{6.8}$$

$$k_c(t) = \frac{p_o(t) h_c}{r_o} \tag{6.9}$$

$$\dot{p}_c(t) = -2a_c p_c(t) + r_c l_c^2(t) - q_c \tag{6.10}$$

$$\dot{p}_o(t) = 2a_c p_o(t) - r_o k_c^2(t) + q_o \tag{6.11}$$

where a, b, h, and m are the unknown true system parameters, and a_c ,b_c ,h_c, m_c, k_c, l_c are the designed (or nominal) system parameters. p_c is the control Riccati term and p_o is the observer covariance term.

Following the same procedure as in Section 4.2.1, the closed-loop system dynamics can be rewritten to emphasize the modelling errors in the following way.

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} (a_c - b_c l_c(t) + d_{ab}(t)) & (b_c l_c(t) + d_b(t)) \\ d(t) & (a_c - k_c(t) h_c + d_{bm}(t)) \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \tag{6.12}$$

where $d_{ab}(t)$, $d_b(t)$, $d(t)$, and $d_{bm}(t)$ are defined in equations (4.14), (4.15), (4.18), and (4.19), respectively. Considering steady-state only, $\dot{p}_c(t)$ and $\dot{p}_o(t)$ can be set to zero in equations (6.10) and (6.11) such that

$$p_c = \frac{r_c l_c^2 - q_c}{2a_c} \tag{6.13}$$

$$p_o = \frac{r_o k_c^2 - q_o}{2a_c} \tag{6.14}$$

6.2.1.2 Steady-State Eigenvalue and Lyapunov Function Analysis

The purpose of this effort is to determine acceptable ranges of modelling errors in a, b, h, and m (i.e. Δa, Δb, Δh, and Δm which are defined in equations (4.9) to (4.12)) for which the system (equation 6.12) remains stable. One way is to look at the eigenvalues of the system matrix, $\bar{a}$, for various modelling errors where

$$\bar{a} = \begin{bmatrix} (a_c - b_c l_c + d_{ab}) & (b_c l_c + d_b) \\ d & (a_c - k_c h_c + d_{bm}) \end{bmatrix} \qquad (6.15)$$

Note that if there were no modelling errors, the stability of the closed loop system (equation 6.12) is determined by the eigenvalues of the closed loop system matrix $(a_c - b_c l_c)$ and the observer system matrix $(a_c - k_c h_c)$, separately [122]. The modelling errors were varied independently, until the real parts of the eigenvalues of $\bar{a}$ in equation (6.15) became positive. This would identify a bound (or range of values) for which the system would remain stable.

This same approach is applied to the Lyapunov functions derived in Sections III and IV. The Lyapunov function in Section III, which is the Lyapunov function for the continuous, linear, time-varying system without parameter uncertainties is presented in equations

(4.1)-(4.5). When parameter uncertainties are introduced in the system model, the Lyapunov function remains positive definite. However, the sufficient condition for $\dot{V}$ to be negative semidefinite is provided in the inequality constraint of equation (4.22). Considering the steady-state scalar problem, the equation becomes

$$-\begin{bmatrix} r_c l_c^2 + q_c & -r_c l_c^2 \\ -r_c l_c^2 & r_c l_c^2 + q_o \end{bmatrix} + \begin{bmatrix} 2d_{ab}p_c & dp_o + d_b p_c \\ dp_o + d_b p_c & 2d_{bm}p_o \end{bmatrix} < 0 \qquad (6.16)$$

For the system to remain stable in the sense of Lyapunov, the eigenvalues of the left side of equation (6.16) must be negative.

For the Lyapunov function derived with parameter uncertainties (eqn. (4.38)), the conditions for stability are different then equation (6.16). For this Lyapunov function, $\dot{V}$ is negative semidefinite for any uncertainty. However, the sufficient condition for V to be positive definite is provided in the inequality constraint of equation (4.40). Considering the steady-state scalar problem, the equation becomes

$$\begin{bmatrix} \Lambda_X & \Lambda_S \\ \Lambda_S^T & \Lambda_P \end{bmatrix} > 0 \qquad (6.18)$$

where

$$\begin{bmatrix} \Lambda_X & \Lambda_S \\ \Lambda_S^T & \Lambda_P \end{bmatrix} \begin{bmatrix} (a_c - b_c l_c + d_{ab}) & (b_c l_c + d_b) \\ d & (a_c - k_c h_c + d_{bm}) \end{bmatrix}$$

$$+ \begin{bmatrix} (a_c - b_c l_c + d_{ab})^T & d^T \\ (b_c l_c + d_b)^T & (a_c - k_c h_c + d_{bm})^T \end{bmatrix} \begin{bmatrix} \Lambda_X & \Lambda_S \\ \Lambda_S^T & \Lambda_P \end{bmatrix}$$

$$+ \begin{bmatrix} r_c l_c^2 + q_c & -r_c l_c^2 \\ -r_c l_c^2 & r_c l_c^2 + q_o \end{bmatrix} = 0 \qquad (6.19)$$

which is an algebraic Lyapunov equation. For the system to remain stable in the sense of Lyapunov, the eigenvalues of the left side of equation (6.18) must be positive.

For the eigenvalue analysis and the two Lyapunov functions, the system parameters were chosen as [122]

$$a_c = 1 \quad , \quad b_c = 1 \quad , \quad h_c = 1 \quad , \quad r_c = 1 \quad ,$$

$$q_c = 1 \quad , \quad r_o = 1 \quad , \quad q_o = 1 \qquad (6.20)$$

For all three cases, the system modelling errors (Δa, Δb, Δh, and Δm) were varied independently until the stability conditions were violated. The results are shown in Table 6.1. By comparing the results of the Lyapunov equations to the eigenvalue analysis (which is known to be valid), a measure of the effectiveness of each Lyapunov function to identify regions of stability

System Matrix Eigenvalues	Lyapunov Function w/o Parameter Uncertainties	Lyapunov Function w/ Parameter Uncertainties
$\Delta a < .39$	$-8.75 < \Delta a < .1$	$\Delta a < .39$
$\Delta b > -.3$	$-.2 < \Delta b < .2$	$\Delta b > -.3$
$\Delta h > -.3$	$-.1 < \Delta h < 2.1$	$\Delta h > -.3$
$\Delta m < .07$	$-.025 < \Delta m < .025$	$\Delta m < .07$

TABLE 6.1 Acceptable Ranges of Parameter Uncertainties for Scalar Problem

can be obtained. Note that the Lyapunov function derived without parameter uncertainties has both upper and lower bounds on the acceptable ranges of parameter uncertainties. This is because of the quadratic nature of the inequality constraint (equation 6.16). The bounds tended to be a little tighter then that of the system eigenvalue analysis. The bounds on the Lyapunov function derived with parameter uncertainties are equivalent to those of the system eigenvalue analysis. These bounds are very similiar to those found by Speyer [122].

The Lyapunov function from Section II, which consisted of combining the separate controller and observer Lyapunov functions, is also evaluated under parameter variations, and was found to be an invalid Lyapunov function given the system parameters in equation (6.20), even without any parameter variations. In Section II, it was pointed out that this particular function (equation (2.67)) can not be analytically shown to be a valid Lyapunov function for the closed-loop system with an observer in the loop. The numerical results have reinforced these analytic statements and have demonstrated that equation (2.67) is not a valid Lyapunov function for all choices of system parameters.

6.2.2 Linear, Time-Invariant Multivariable Control Problem

The linear, time-invariant, multivariable control problem was selected from an example by Doyle and Stein [41] and is as follows

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 35 \\ -61 \end{bmatrix} V \qquad (6.21)$$

$$y = [2 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \omega \qquad (6.22)$$

$$E[V(t) V^T(\tau)] = \delta(t - \tau) \qquad (6.23)$$

$$E[\omega(t)\omega^T(\tau)] = \delta(t - \tau) \tag{6.24}$$

$$E[v(t)\omega^T(\tau)] = 0 \tag{6.25}$$

$$u = -[50 \quad 10]\begin{bmatrix}\hat{x}_1 \\ \hat{x}_2\end{bmatrix} \tag{6.26}$$

$$\begin{bmatrix}\dot{\hat{x}}_1 \\ \dot{\hat{x}}_2\end{bmatrix} = \dot{\hat{x}} = (A-BL)\hat{x} + K[y-H\hat{x}] \tag{6.27}$$

$$\dot{P}_c = -P_cA - A^TP_c + L^TR_cL + Q_c \tag{6.28}$$

$$\dot{P}_o = AP_o + P_oA^T - KR_oK^T + Q_o \tag{6.29}$$

$$K = P_oH^TR_o^{-1} \tag{6.30}$$

where

$$R_c = 1 \tag{6.31}$$

$$Q_c = \begin{bmatrix}2800 & 473.29 \\ 473.29 & 80\end{bmatrix} \tag{6.32}$$

$$R_o = 1 \tag{6.33}$$

$$Q_o = \begin{bmatrix}1225 & -2135 \\ -2135 & 3721\end{bmatrix} + q^2\begin{bmatrix}0 & 0 \\ 0 & 1\end{bmatrix} \tag{6.34}$$

This system represents a weakly stable system, where weakly implies the system has poor phase margin [41]. Doyle and Stein set out to increase the stability (or robustness) of the system by adding a constant fictitious term, q^2, to the process noise covariance matrix, Q_o. By using Nyquist diagrams, Doyle and Stein were able to come up with a reasonable compromise between noise performance and robustness by increasing q^2 [41]. By increasing q^2, the error covariance increases and the closed loop stability margins improve.

6.2.2.1 Steady-State Eigenvalue and Lyapunov Function Analysis

As in the scalar case in Section 6.2.1.2, steady-state analysis is applied to the closed loop system and the two Lyapunov functions. For this example, a variation in the control matrix, B, is investigated (i.e. ΔB). Acceptable ranges of ΔB variations were generated for q^2 set to 0, 100, 1000, and 10000, which were the same values selected by Doyle and Stein.

For the eigenvalue analysis, the closed loop system matrix is

$$\begin{bmatrix} A_c - B_c L_c + D_{AB} & B_c L_c + D_B \\ D & A_c - K_c H_c + D_B \end{bmatrix} \tag{6.35}$$

where D_{AB}, D_B, and D come from equations (4.14), (4.15), and (4.18), respectively. The inequality constraint that validates the Lyapunov function derived without parameter variations is equation (4.26) and the inequality constraint that validates the Lyapunov function derived with parameter variations is equation (4.40). The results are shown in Table 6.2.

q^2	System Matrix Eigenvalues	Lyapunov Function w/o Parameter Uncertainties	Lyapunov Function w/ Parameter Uncertainties
0	$\Delta B > -.2$	$-.0025 < \Delta B < .0025$	$\Delta B > -.2$
100	$\Delta B > -.25$	$-.0045 < \Delta B < .0045$	$\Delta B > -.25$
1000	$\Delta B > -.65$	$-.0155 < \Delta B < .017$	$\Delta B > -.65$
10000	$\Delta B > -1.05$	$-.0285 < \Delta B < .0275$	$\Delta B > -1.05$

TABLE 6.2 Acceptable Ranges of Parameter Uncertainties for Multivariable Problem.

The Lyapunov function derived without parameter uncertainties has both upper and lower bounds on ΔB, as in the scalar case. These bounds are much narrower than in the scalar problem. The bounds on ΔB produced by the Lyapunov function which includes parameter uncertainties are identical to those of the system eigenvalue analysis. As in the paper by Doyle and Stein [41], the system stability margins increased as q^2 is increased in both Lyapunov function analysis. The Lyapunov function, which consisted of combining the separate controller and observer Lyapunov functions, was also evaluated under variations of ΔB, and was found to be an invalid Lyapunov function given the system parameters defined in the paper [41].

6.2.2.2 Performance Analysis Through the Lyapunov Equation

In the previous section the Lyapunov function derived with parameter uncertainties is very accurate in identifying acceptable ranges of ΔB variations for system stability through steady-state analysis. In this section, the actual time response of the Lyapunov equation (4.80) for the Lyapunov function derived with parameter uncertainties is investigated.

Figures 6.1 to 6.4 show the minimum eigenvalue of the Lyapunov equation for q^2=0, 100, 1000, and 10000. For each value of q^2, several values of ΔB are considered. The figures do not show any significant changes to the minimum eigenvalues. The figures do show that the solution to the Lyapunov equation is positive definite for K>0.

Figures 6.5 to 6.8 show the maximum eigenvalue of the Lyapunov equation for the same values of q^2 and ΔB. When ΔB exceeds the acceptable ranges identified in the previous section, the maximum eigenvalue becomes unbounded with very large negative slopes. With the system going unstable and the states diverging, the Lyapunov equation becomes unbounded from above. Since the steady-state analysis is meaningless for the time-varying problem, this type of analysis is more appropriate.

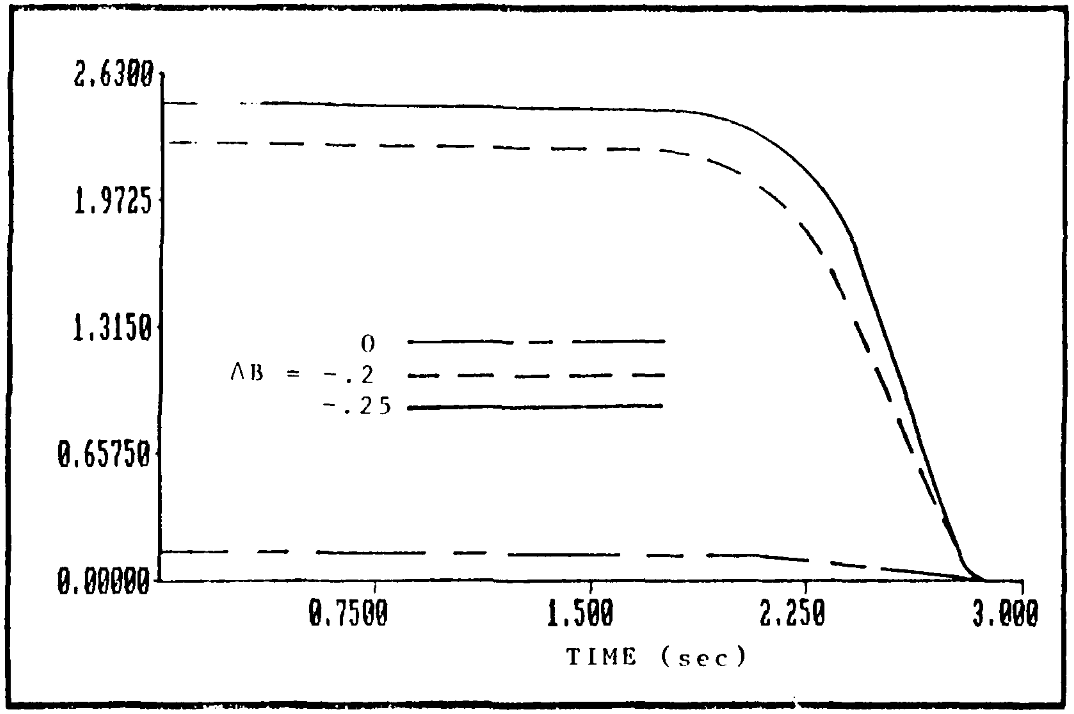


Figure 6.1, Minimum Eigenvalue of the Lyapunov Equation for the Lyapunov Function Derived with Parameter Uncertainties, $q^2=0$.

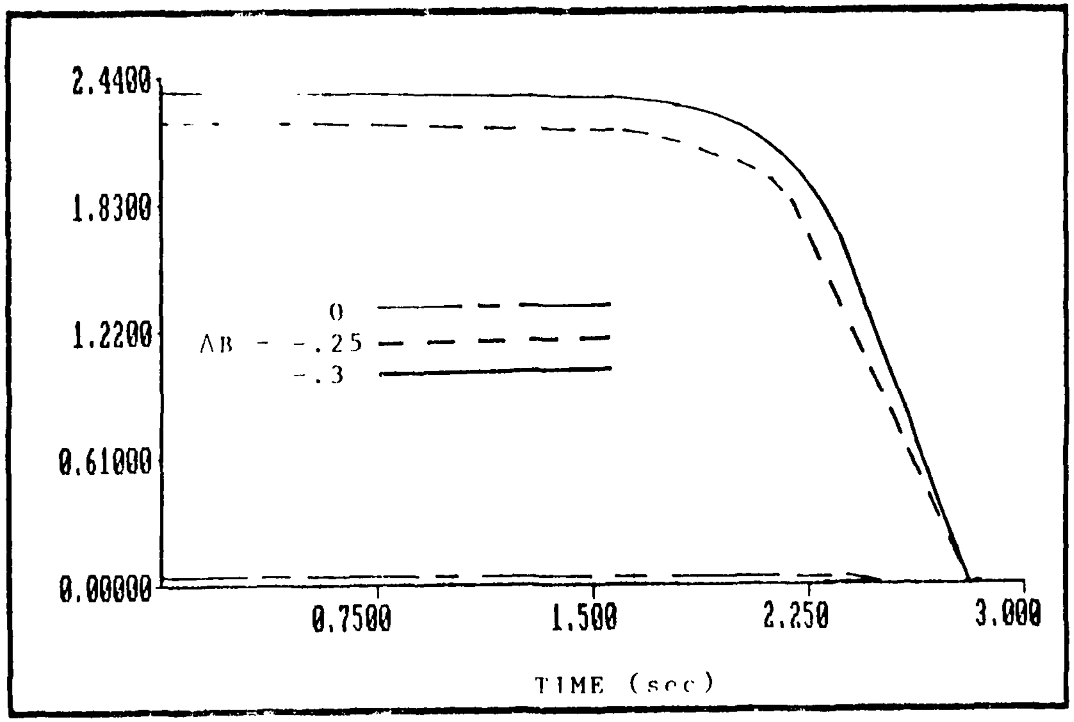


Figure 6.2, Minimum Eigenvalue of the Lyapunov Equation for the Lyapunov Function Derived with Parameter Uncertainties, $q^2=100$.

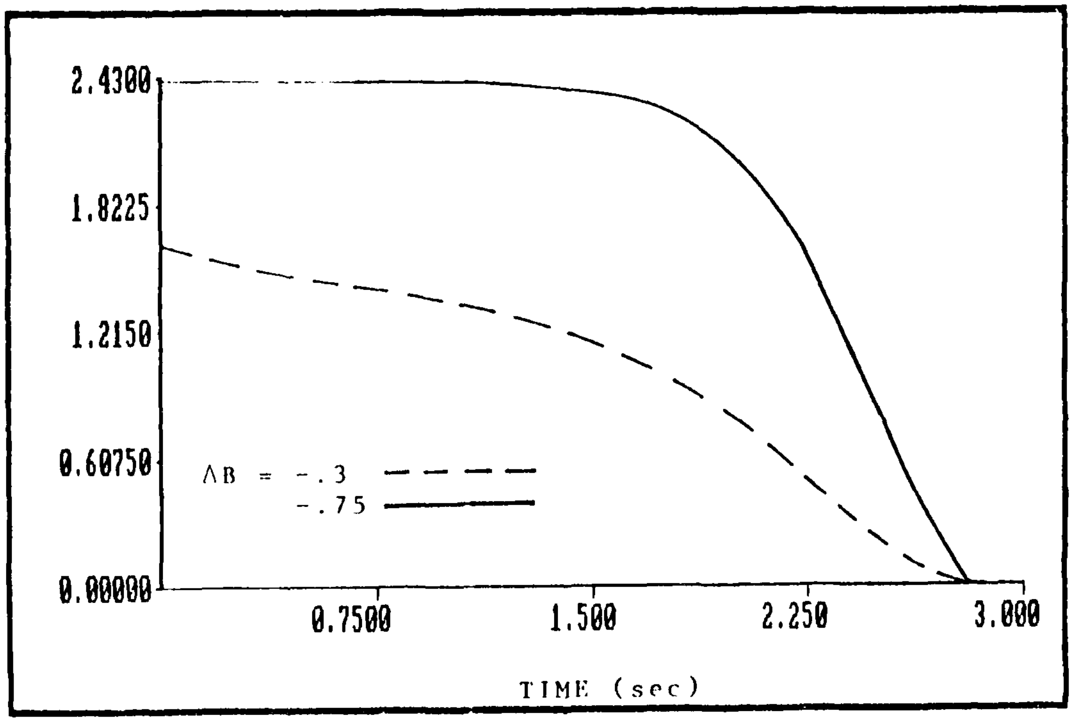


Figure 6.3, Minimum Eigenvalue of the Lyapunov Equation for the Lyapunov Function Derived with Parameter Uncertainties, $q^2 = 1000$.

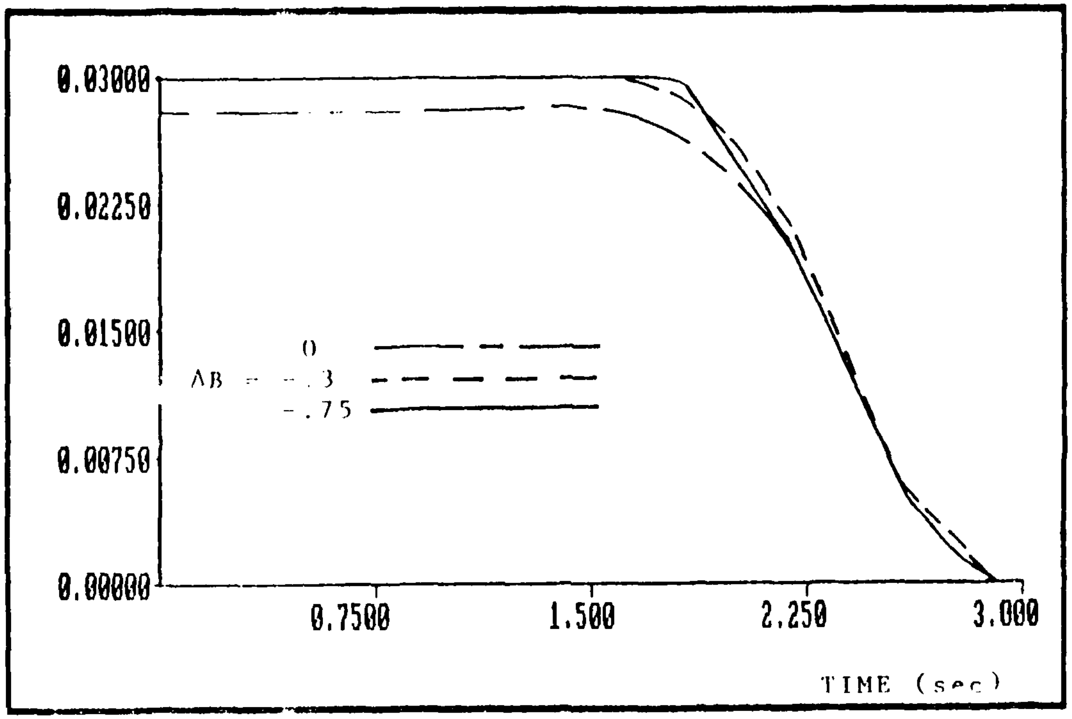
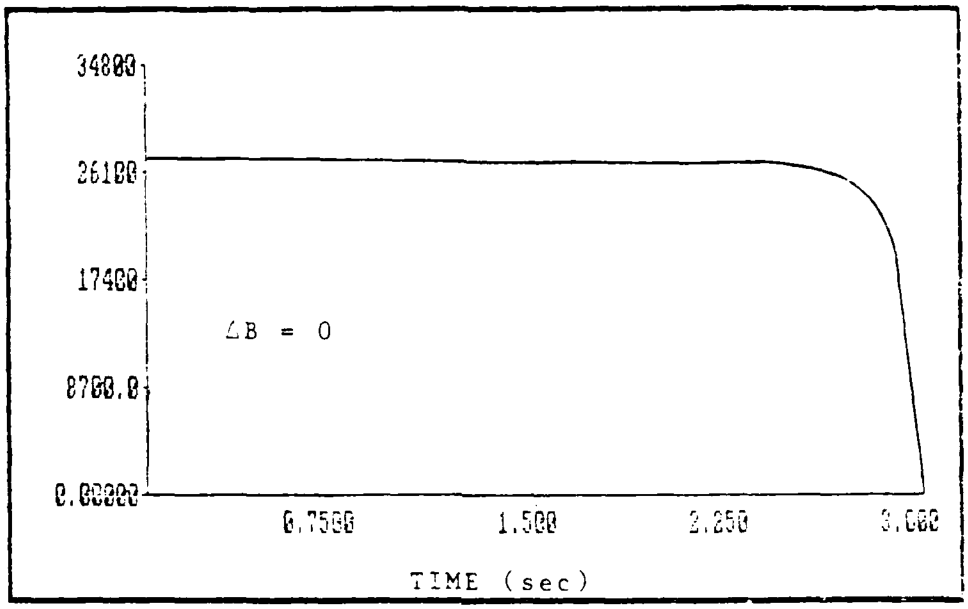


Figure 6.4, Minimum Eigenvalue of the Lyapunov Equation for the Lyapunov Function Derived with Parameter Uncertainties, $q^2=10000$.



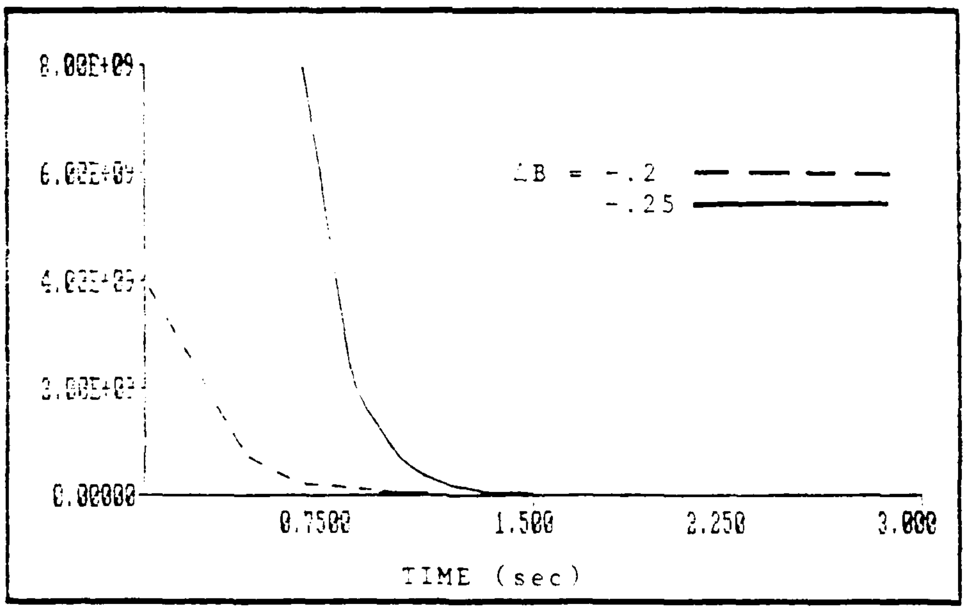


Figure 6.5, Maximum Eigenvalue of the Lyapunov Equation for the Lyapunov Function Derived with Parameter Uncertainties, $q^{-}=0$.

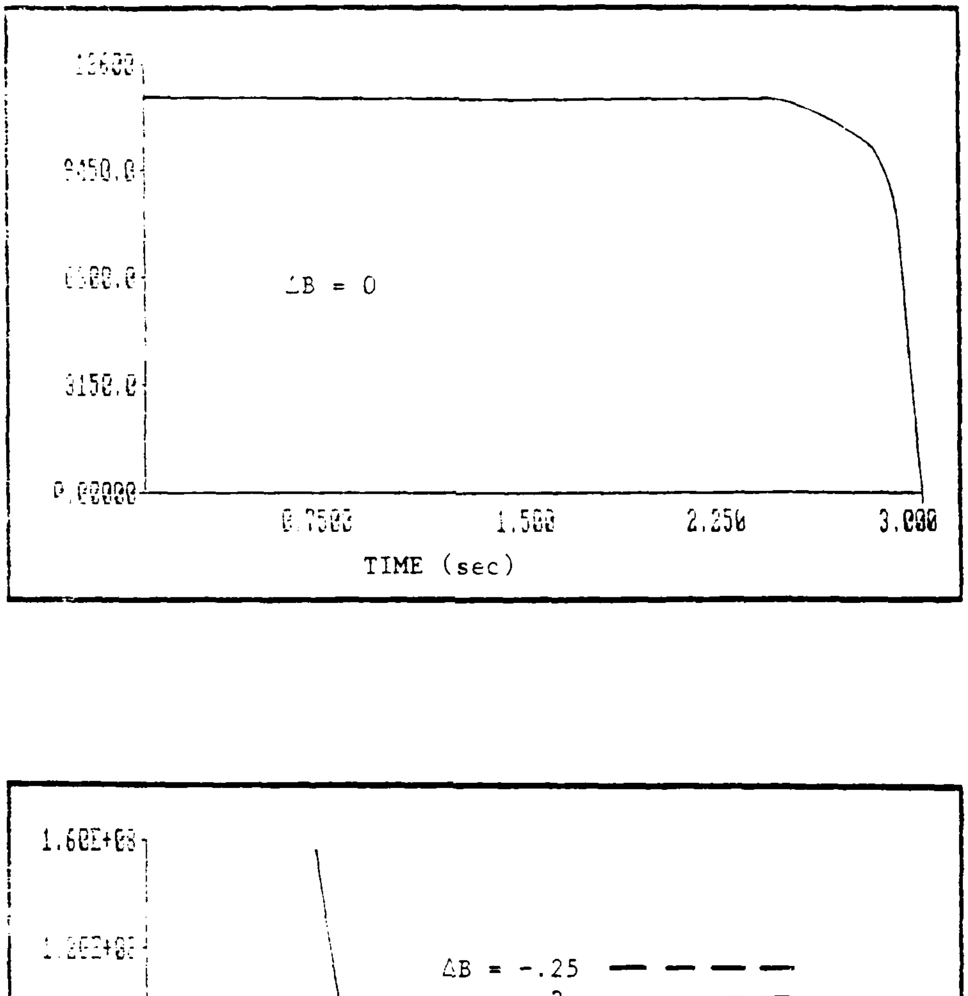


Figure 6.6, Maximum Eigenvalue of the Lyapunov Equation for the Lyapunov Function Derived with Parameter Uncertainties, $q^2=100$.

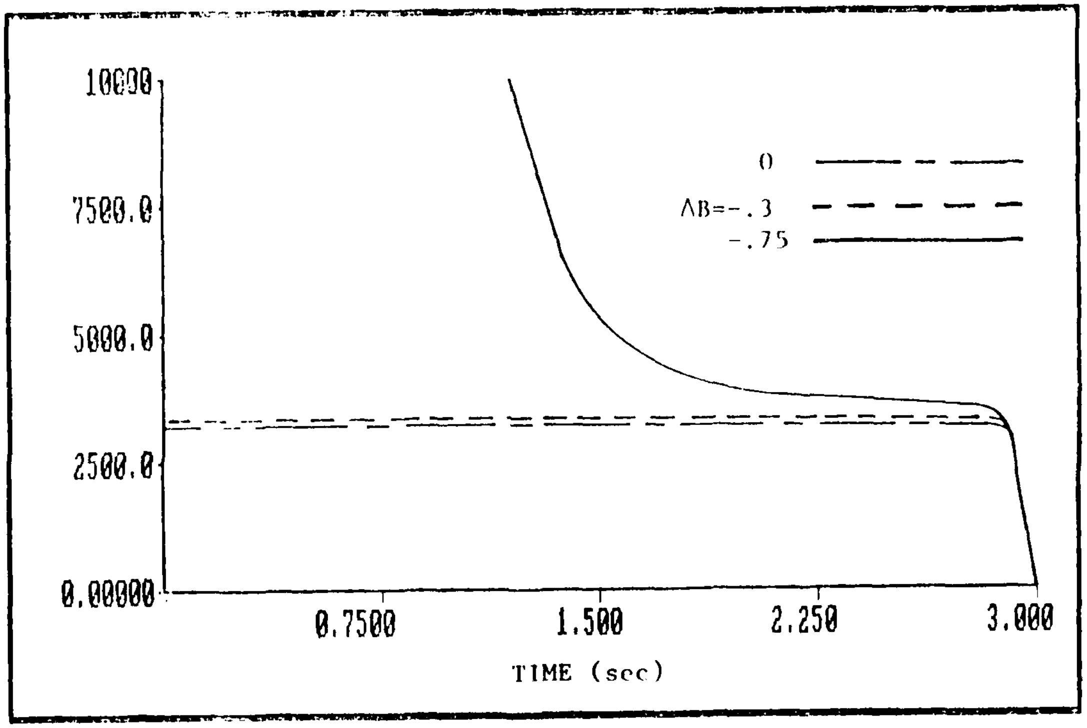


Figure 6.7, Maximum Eigenvalue of the Lyapunov Equation for the Lyapunov Function Derived with Parameter Uncertainties, q^2=1000.

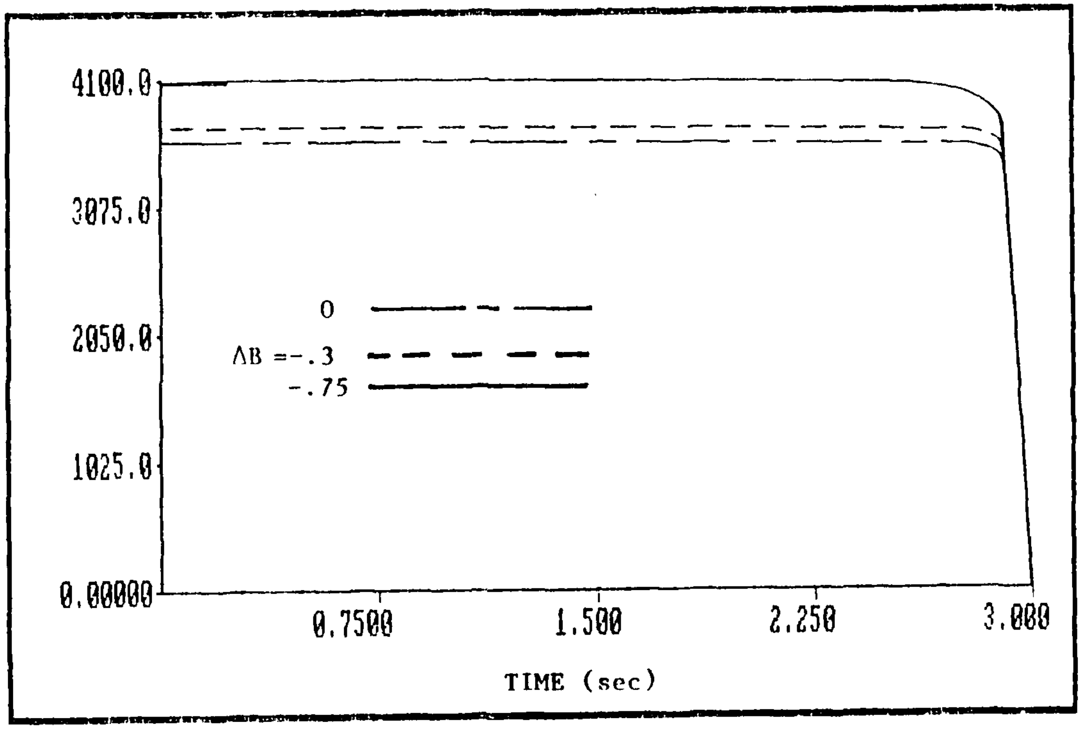


Figure 6.8, Maximum Eigenvalues of the Lyapunov Equation for the Lyapunov Function Derived with Parameter Uncertainties, $q^2=10000$.

6.2.2.3 Robustness Improvements Through Lyapunov Function

For this linear problem, Doyle and Stein improved the robustness characteristics (or stability margin) by adding a "fictitious noise" term to the process noise covariance matrix [41]. A more systematic way to improve stability margins might be available by taking a closer look at the Lyapunov function which consists of combining the separate controller and observer Lyapunov functions (equation 2.67).

From Section II, the condition necessary for this function to be a valid Lyapunov function is that

$$\begin{bmatrix} -L^TRL-Q_c & L^TRL \\ L^TRL & -P^{-1}Q_oP^{-1}-H^TR_o^{-1}H \end{bmatrix} < 0 \qquad (6.36)$$

This may not always be true (and in the cases, so far, it has not been true) ; however, R, Q_c, R_o, and Q_o can be choosen to ensure that the inequality constraint is valid. Doyle and Stein's approach involved increasing Q_o, which would improve the negative definiteness of the left side of equation (6.36). Another way might be to decrease R_o. This implies that, given the controller, the measurement device has to have a certain accuracy to ensure the system remains stable. A third way would be to decrease Q_c in the control design. This has the effect of decreasing the control gain, L. Although this can make the system more robust, it has the adverse effect of reducing the response of the closed loop system. There is a tradeoff to be made between system response time and robustness to system modelling errors.

For this study, Q_c was changed to $\overline{Q}_c$ by the following:

$$\overline{Q}_c = \alpha Q_c \tag{6.37}$$

where

$$\alpha = .001 \tag{6.38}$$

This changed the control gain to

$$L = [\ .425 \quad .116\ ] \tag{6.39}$$

and satisfied the inequality constraint, equation (6.42).

The closed loop simulation was run using this new control gain for $q^2 = 0$ and $\Delta B = 0$, -.25, and -.75. Only $q^2 = 0$ was used since it demonstrated the least system robustness properties. Figures 6.9, 6.10, and 6.11 represent the system by Doyle and Stein using the control gain from equation (6.26). Figures 6.12, 6.13, and 6.14 represent the same system with the exception of the control gain, L, (equation 6.39). The results on the last three figures show that the system is much more robust then the results using the original control gain; although the system is somewhat less responsive.

Thus, this Lyapunov function provides a means for making the controller/observer system more stable (or robust) through an overall design selection of the controller and observer parameters.

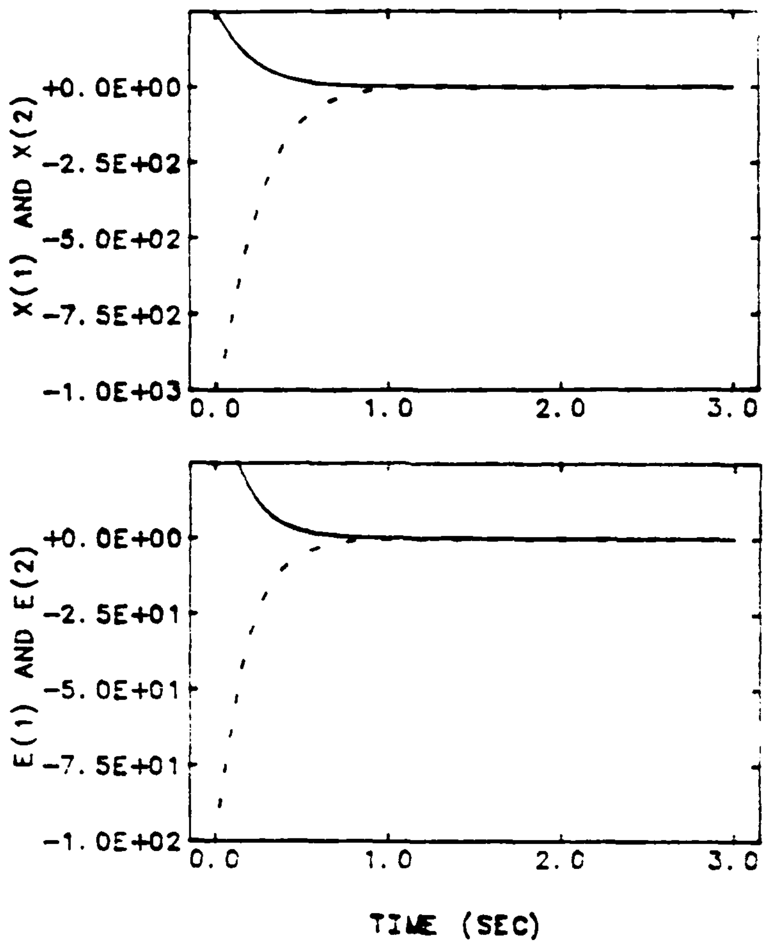


Figure 6.9, $\sigma^2 = 0$, $\underline{\Delta P} = 0$

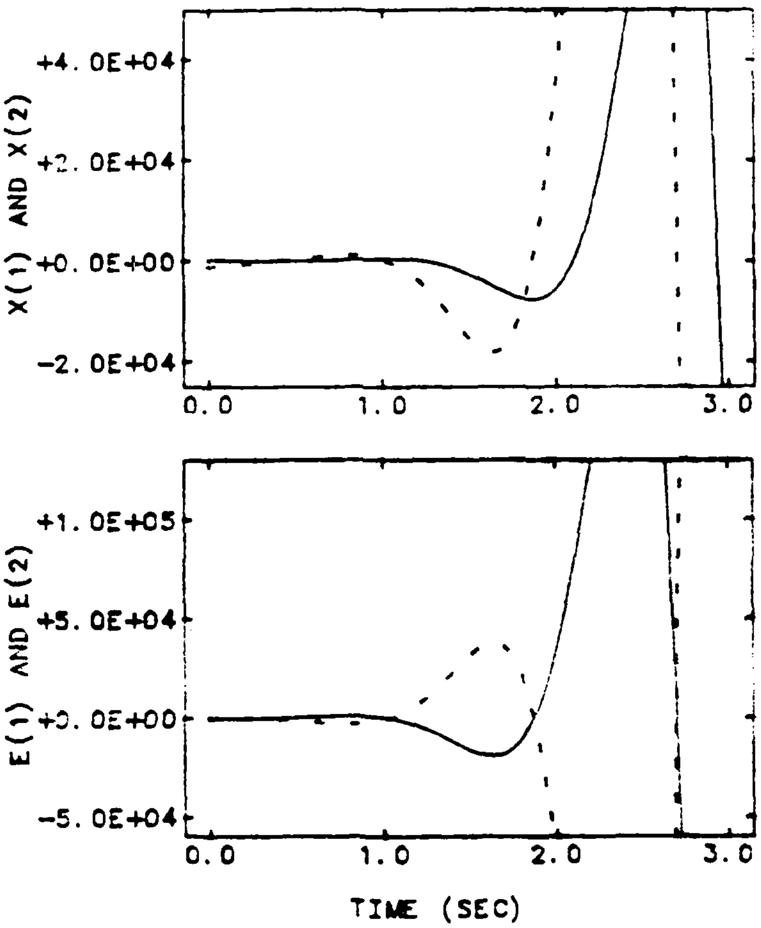


Figure 6.10, $\sigma^2 = 0$, $\underline{\Delta B} = -.25$

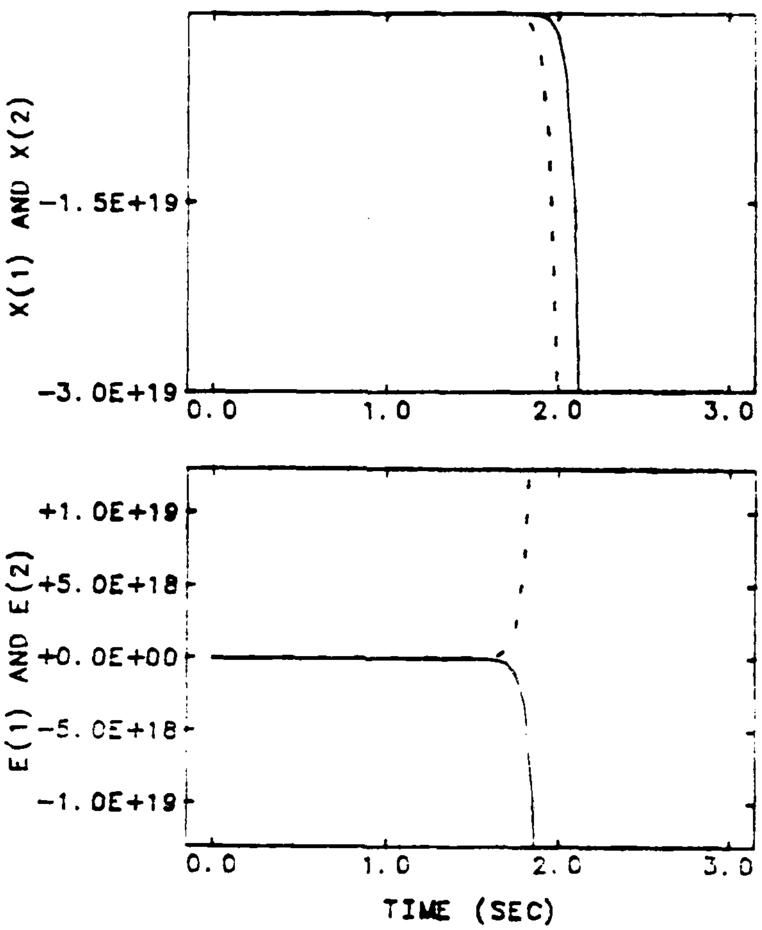


Figure 6.11, $c^2 = 0$, $\underline{\Delta P} = -.75$

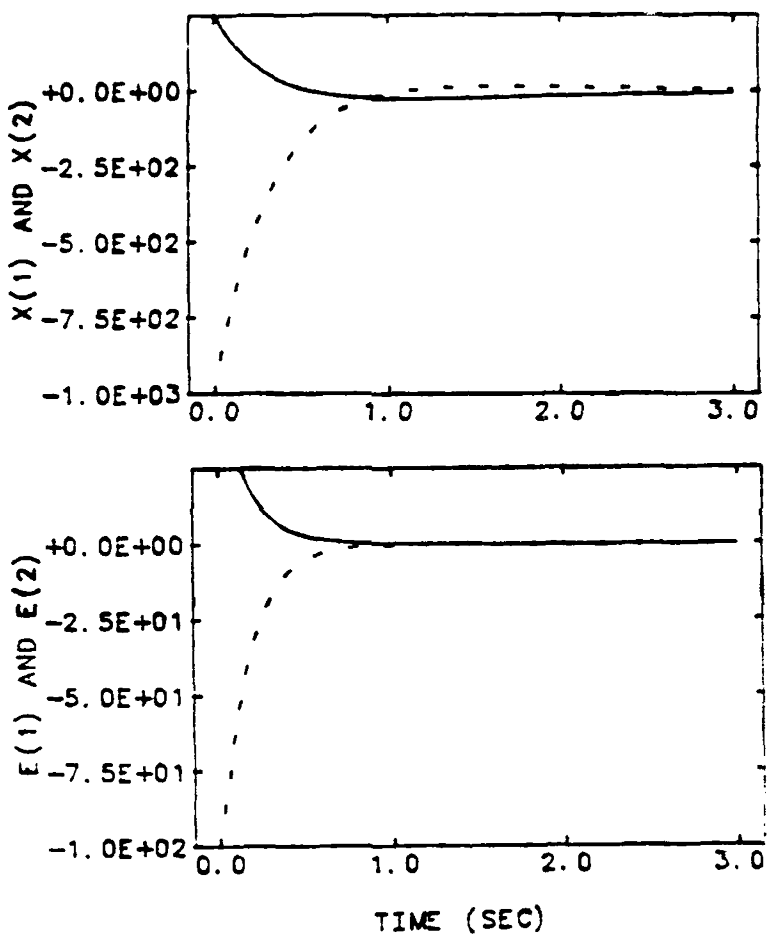


Figure 6.12, Control Gain From Lyap.
Analysis , $\sigma^2 = 0$, $\Delta P = 0$

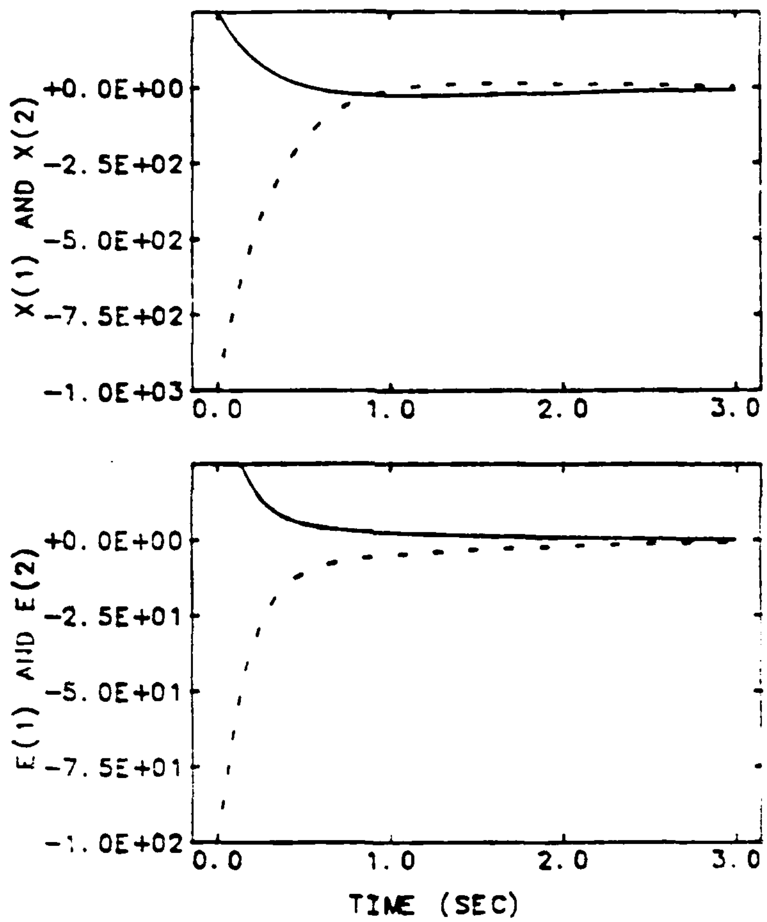


Figure 6.13, Control Gain From Lyap. Analysis , $\sigma^2 = 0$, $\underline{\Delta}P = -.25$

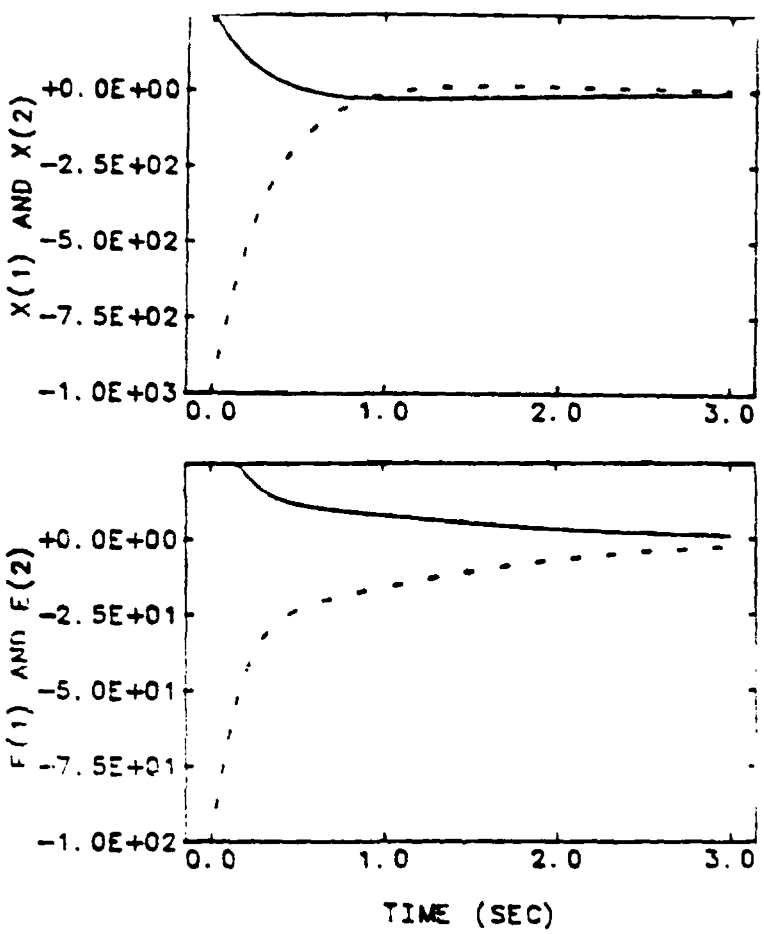


Figure 6.14, Control Gain From Lyap. Analysis, $\sigma^2 = 0$, $\underline{\Delta P} = -.75$

6.3 Linear, Time-Varying Guidance Problem

The system selected for this analysis is

$$\dot{x} = Ax + Bu(t) + \omega \ , \ x(t_o) = x_o \qquad (6.40)$$

where x is a 6-state vector of the 2-dimensional components of relative position, velocity, and target acceleration, and u is the 2-dimensional missile acceleration. In addition,

$$A = \begin{bmatrix} \emptyset & I & \emptyset \\ \emptyset & \emptyset & I \\ \emptyset & \emptyset & -\lambda_T I \end{bmatrix} \qquad (6.41)$$

$$B = \begin{bmatrix} \emptyset \\ -I \\ \emptyset \end{bmatrix} \qquad (6.42)$$

$$u(t) = -L(t)\hat{x} \qquad (6.43)$$

$$\dot{\hat{x}} = (A-BL(t))\hat{x} + K[y-H\hat{x}] \ , \ \hat{x}(t_o) = \hat{x}_o \qquad (6.44)$$

$$y = Hx + \nu \qquad (6.45)$$

$$H = [\ I \quad \emptyset \quad \emptyset\ ] \qquad (6.46)$$

$$\omega \sim N(\emptyset, Q_o) \quad , \quad \nu \sim N(\emptyset, R_o) \qquad (6.47)$$

$$Q_o = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & .1I \end{bmatrix} \quad (6.48)$$

$$R_o = \begin{bmatrix} .01 & 0 \\ 0 & .01 \end{bmatrix} \quad (6.49)$$

where λ^T is the target acceleration response time coefficient and I is a 2x2 identity matrix. The initial conditions for the closed loop system are

$$x_o = [\ 3500\ ,\ 1500\ ,\ -1100\ ,\ -150\ ,\ 10\ ,\ 10\ ]^T \quad (6.50)$$

$$\hat{x}_o = [\ 3000\ ,\ 1200\ ,\ -950\ ,\ -100\ ,\ 0\ ,\ 0\ ]^T \quad (6.51)$$

6.3.1 Time-Varying Guidance Law

The guidance law selected for this study comes from linear quadratic Gaussian theory, and is derived from the following optimization problem [109]

$$J = \frac{1}{2}x_f^T G_f x_f + \frac{1}{2}\int_0^{t_f} u^T R u \, dt \quad (6.52)$$

subject to equation (6.40), where

$$G_f = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{6 \times 6} \quad (6.53)$$

$$R = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} , \quad b = 10^{-4} \quad (6.54)$$

This cost functional is constructed to minimize final miss distance with no weighting on final relative velocity nor target acceleration, and a weighted cost on the control (missile acceleration) through the integral term [109]. The weighting factor, b, determines the degree of cost of control versus cost of terminal miss distance. A small value of b implies more emphasis is placed on minimizing terminal miss distance at the cost of control effort.

An important point to make is that the optimization problem is based on the assumption that the control vector, u, is the missile's acceleration. This implies that the missile has instantaneous response and complete control over all inertial acceleration components.

The optimization problem generates a linear, closed-form control law of the form [109]

$$u(t) = -\frac{3t_g}{3b+t_g}[\; I \quad t_g I \quad K_T I \;]x(t) \qquad (6.55)$$

where

$$t_g = t_f - t \qquad (6.56)$$

and

$$K_T = (e^{-\lambda_T t_g} + \lambda_T t_g - 1)/\lambda_T^2 \qquad (6.57)$$

In practice, the control law, u(t), is a function of the estimated states, $\hat{x}(t)$, and not the true states, x(t), which are typically unknown. This is justified through the separation principle.

6.3.2 Time-To-Go Error Analysis

Note that the control law, equation (6.55), is an explicit function of t_g. The theory that is used to obtain the solution assumed that the final time, t_f, is specified; therefore, to insure optimality, t_f must be known apriori or at least accurately estimated during flight [109]. Since t_f cannot be realistically known apriori (especially for a maneuvering target) , t_f (or t_g) must be estimated. Studies have shown that the accuracy of t_g can drastically affect the performance of the control law [109,131]; however, what effect does this have on the performance of the system?

To analyze the effects of errors in t_g on the performance of the closed loop system, consider modelling t_g as the following

$$\hat{t}_g = \Sigma\, t_g + B \tag{6.58}$$

where Σ is a scale factor error, B is a bias error and t_g is the true time-to-go, which comes from equation (6.56), where t_f is set to 4 seconds. Scale factor errors are selected as 1.5, 1.0, and 0.5. The effects of these errors are evaluated separately from the bias errors, which are selected as -0.2, 0, and 0.2.

Using equation (6.58) in the control law, a simulation of the system defined in Section 6.3 was run for the various scale factor and bias errors ($\Sigma = 1$ and $\beta = 0$ implies zero errors). The simulation is used to evaluate the three Lyapunov functions derived earlier: The Lyapunov function which is the combination of the seperate controller and observer Lyapunov functions, the Lyapunov function derived without parameter uncertainties, and the Lyapunov function derived with parameter uncertainties. For the different values of scale factor error and bias error, the three Lyapunov functions are checked to determine if they remain a valid Lyapunov function. The combined Lyapunov function is positive definite for all $x,e \neq 0$ and all scale factor and bias errors. The condition for the slope of the Lyapunov function to be negative semidefinite is

$$- \begin{bmatrix} L_{c_K}^T R_K L_{c_K} + Q_{c_K} & -L_{c_K}^T R_K L_{c_K} \\ -L_{c_K}^T R_K L_{c_K} & Q_{e_K} \end{bmatrix}$$

$$+ \begin{bmatrix} D_{AB_K}^T & D_K^T \\ D_{B_K}^T & D_{BM_K}^T \end{bmatrix} \begin{bmatrix} \hat{X}_{K+1} & \emptyset \\ \emptyset & \hat{P}_{K+1} \end{bmatrix} \begin{bmatrix} \bar{A}_K' & B_K' \\ D_K & A_K' \end{bmatrix}$$

$$+ \begin{bmatrix} \bar{A}_K'^T & D_K^T \\ B_K'^T & A_K'^T \end{bmatrix} \begin{bmatrix} \hat{X}_{K+1} & \emptyset \\ \emptyset & \hat{P}_{K+1} \end{bmatrix} \begin{bmatrix} D_{AB_K} & D_{B_K} \\ D_K & D_{BM_K} \end{bmatrix} \leq \emptyset \quad (6.59)$$

The Lyapunov function derived without parameter uncertainties is also positive definite for all $x,e \neq \emptyset$ and all scale factor and bias errors. The condition for the slope of the combined Lyapunov function to be negative semidefinite for all x and e comes from equation (4.65). The Lyapunov function derived with parameter uncertainties is positive definite for all $x,e = \emptyset$ when equation (4.82) is satisfied. The slope of this Lyapunov function is negative semidefinite for all values of scale factor and bias errors.

Since the system evaluated is a finite-time problem, the Lyapunov functions cannot be used as a measure of system stability. The Lyapunov functions are used to provide a measure of system performance. The question is which Lyapunov function is the better one for measuring system performance? Figure 6.15 is a plot of the maximum eigenvalue of equation (6.59) for the combined Lyapunov function, given no scale factor or bias errors. Since this shows equation (6.59) is not negative semidefinite even for the error-free case, the combined Lyapunov function is not a good measure of performance for the system considered. The result is simi- liar to the results found in the steady-state analysis. The maximum eigenvalue for equation (4.65) starts at zero for $t=t_f$ and remains approximately -10^{-12} for $0 \leq t < t_f$ and for all values of scale factor and bias errors. This indicates that the Lyapunov function derived without parameter uncertainties is valid for all time-to-go errors. Figures 6.16 and 6.17 are plots of the minimum eigenvalue of the Lyapunov equation (equation (4.82)) for the Lyapunov function derived with parameter uncertainties, given scale factor and bias errors. The eigenvalues remained positive for $0 \leq t < t_f$, indicating that this Lyapunov function is valid for all time-to-go errors. Figures 6.18 and 6.19 show the maximum eigenvalue of the same Lyapunov equation, given scale factor and bias errors. These eigenvalues remain bounded for all values of scale factor and bias errors.

The results of the last two Lyapunov functions indicate that errors in time-to-go do not degrade the performance of the system for $0 \leq t < t_f$. Figures 6.20-

6.31 are plots of the magnitude of relative position, velocity, and target acceleration; as well as their errors from the estimation algorithm for the set of launch conditions specified in Section 6.3. These results are useful in showing that the combined Lyapunov function is a poor measure of performance for this system with time-to-go errors.

Both the scale factor errors and bias errors do not degrade the performance of the system. An error in t_g basically meant that final time, t_f, was in error. Since t_f is considered a known parameter in the guidance law derivation, the result is that a bias error in t_g will cause the relative range to go to zero at whatever the value of t_f happens to be. For instance, if $B = -.2$ and $t_f = 4$, the estimate of t_f becomes 3.8 seconds. Therefore, as seen in Figure 6.20, range goes to zero at 3.8 seconds. It is irrelevant what happens between 3.8 and 4.0 seconds, since the objective was to drive the range value to zero. Scale factor errors only affect the rate at which the range value converges to zero. Note that because this is a time-varying, linear system, an eigenvalue analysis cannot be performed.

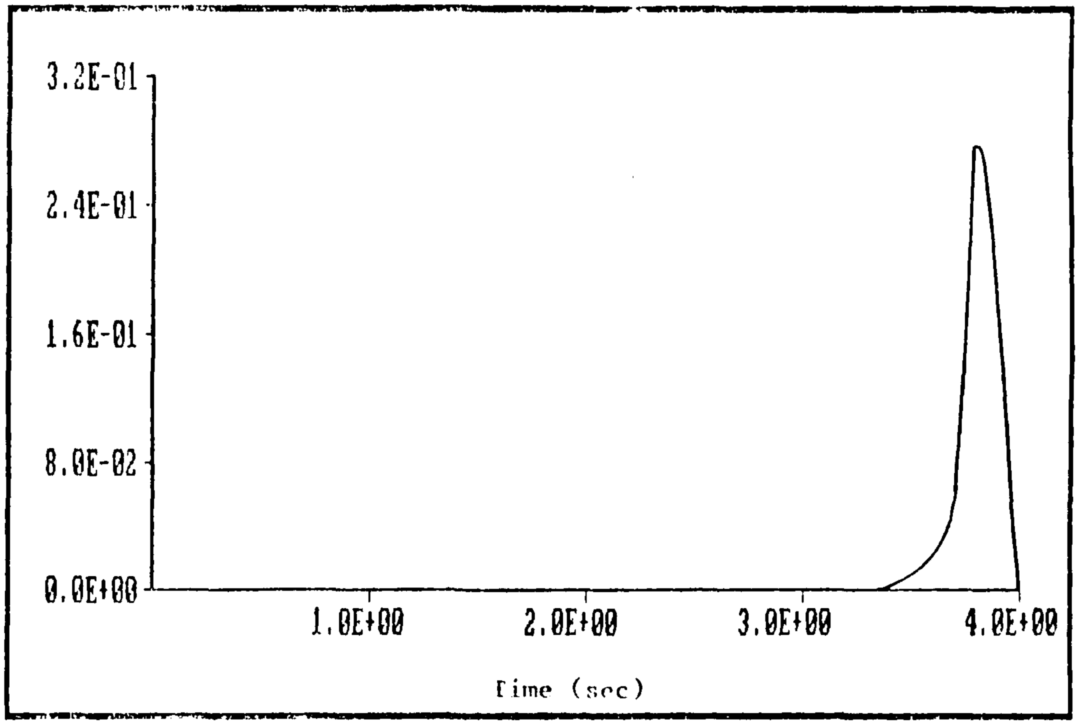


Figure 6.15, Maximum eigenvalue of equation (6.59) for the combined Lyapunov functions $\Sigma = 1, \beta = 0$

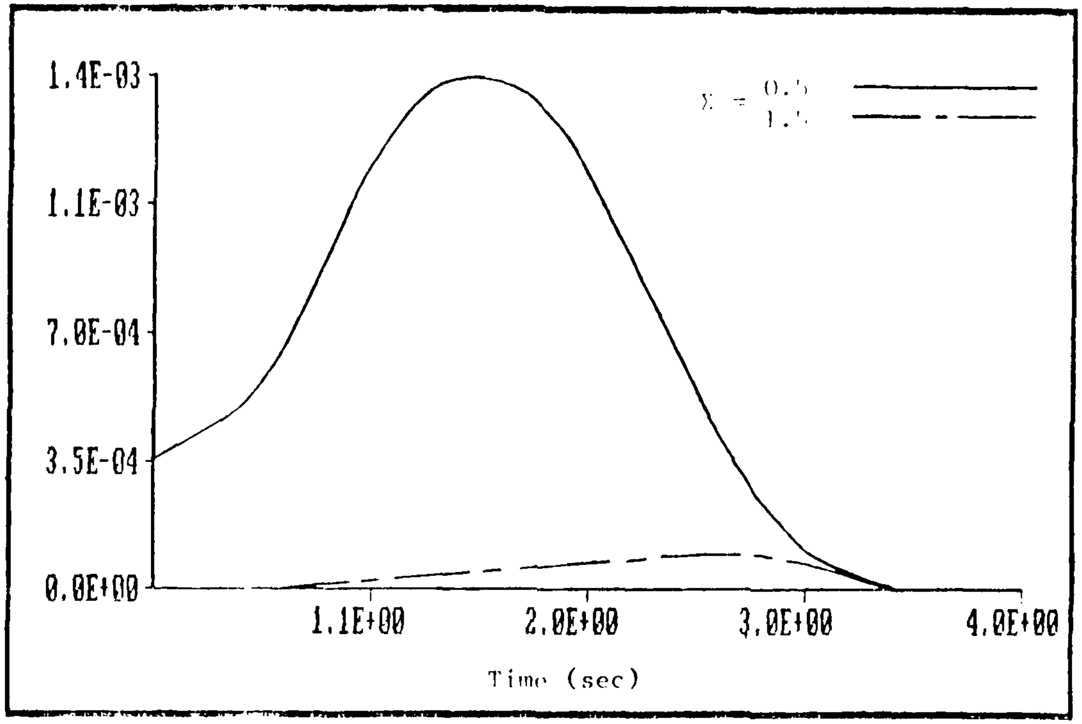


Figure 6.16, Minimum eigenvalue of the Lyapunov equation for the Lyapunov function derived with uncertainties

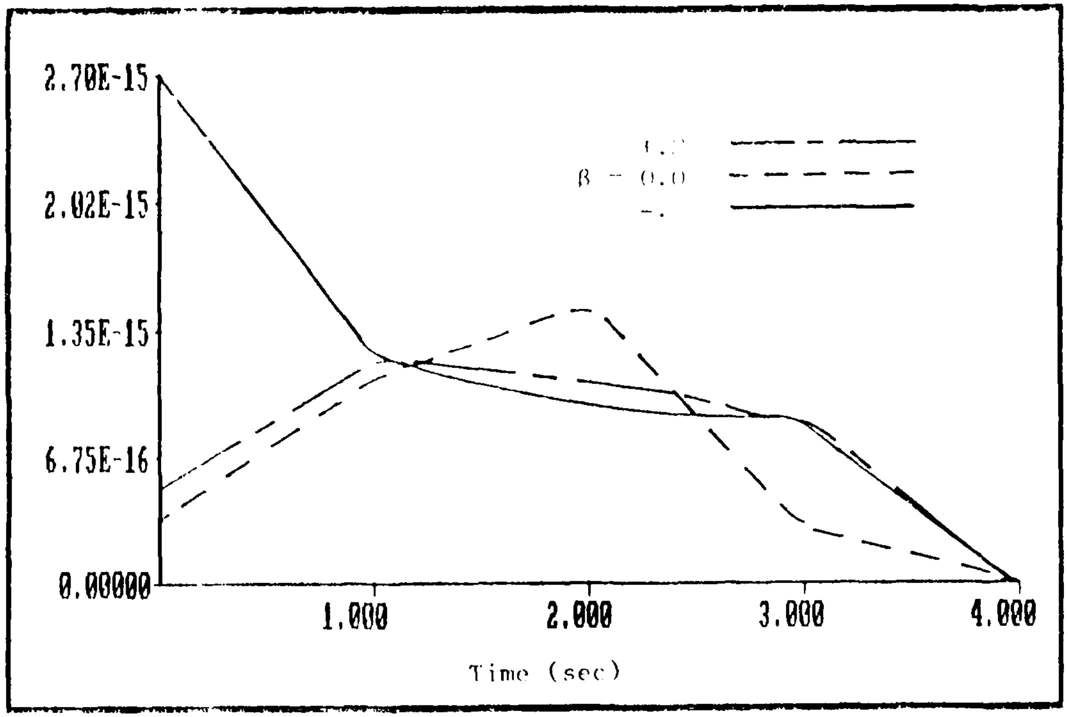


Figure 6.17, Minimum eigenvalue of the Lyapunov equation for the Lyapunov function derived with parameter variations

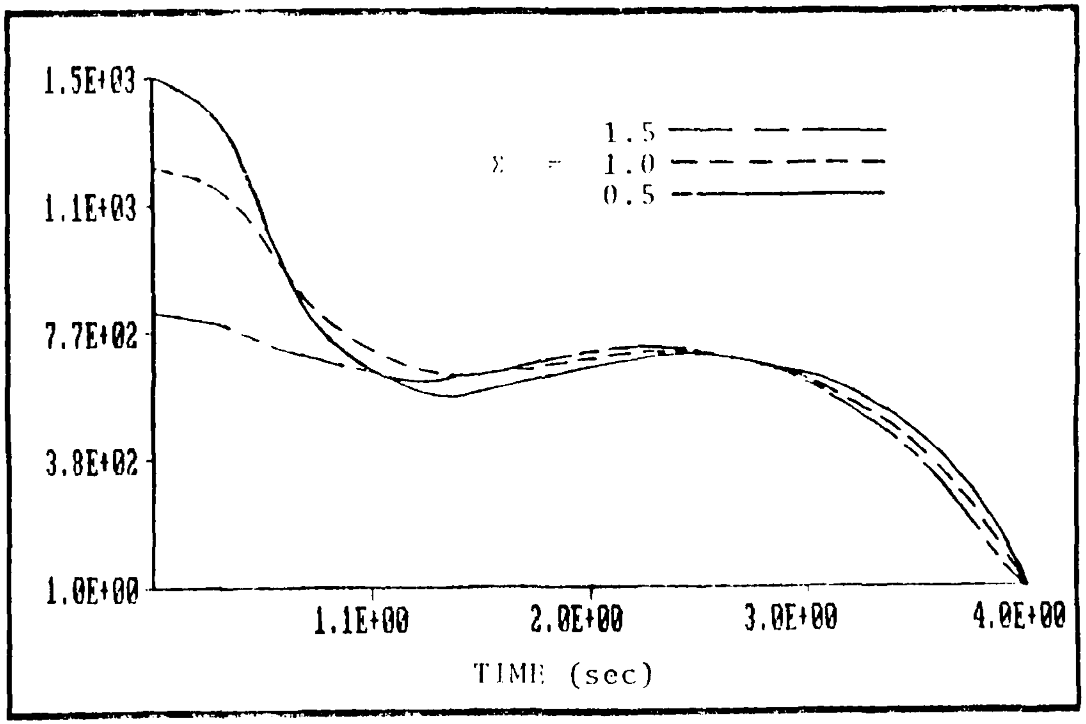


Figure 6.18, Maximum Eigenvalues of the Lyapunov Equation for the Lyapunov Function Derived with Parameter Uncertainties

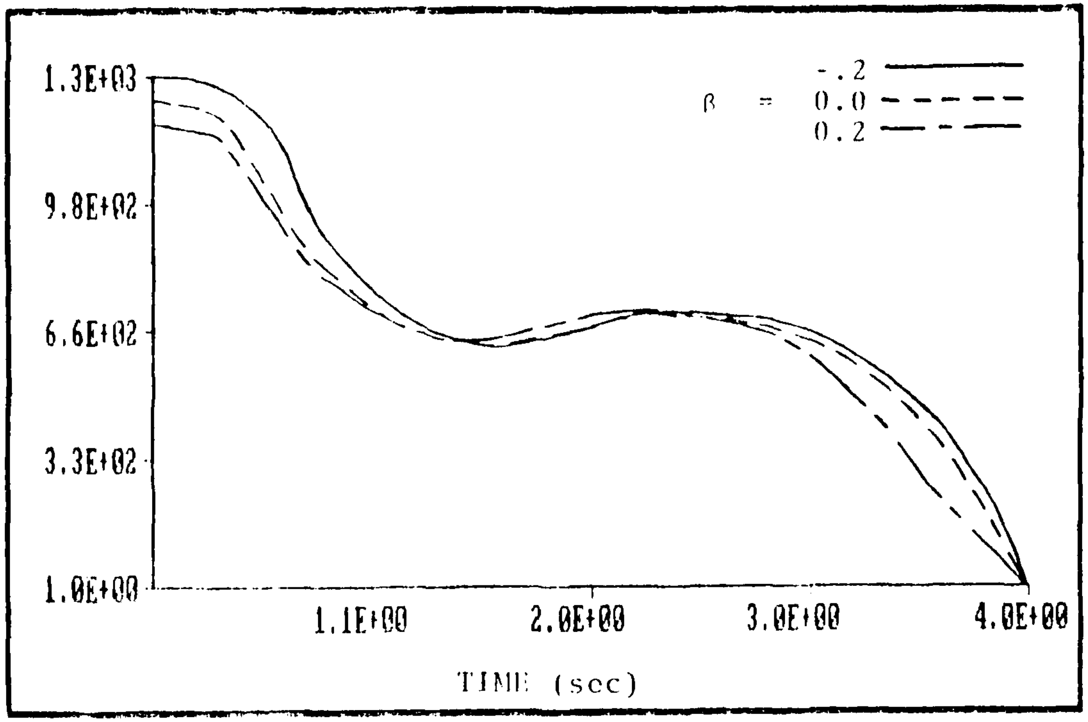


Figure 6.19, Maximum Eigenvalues of the Lyapunov Equation for the Lyapunov Function Derived with Parameter Uncertainties

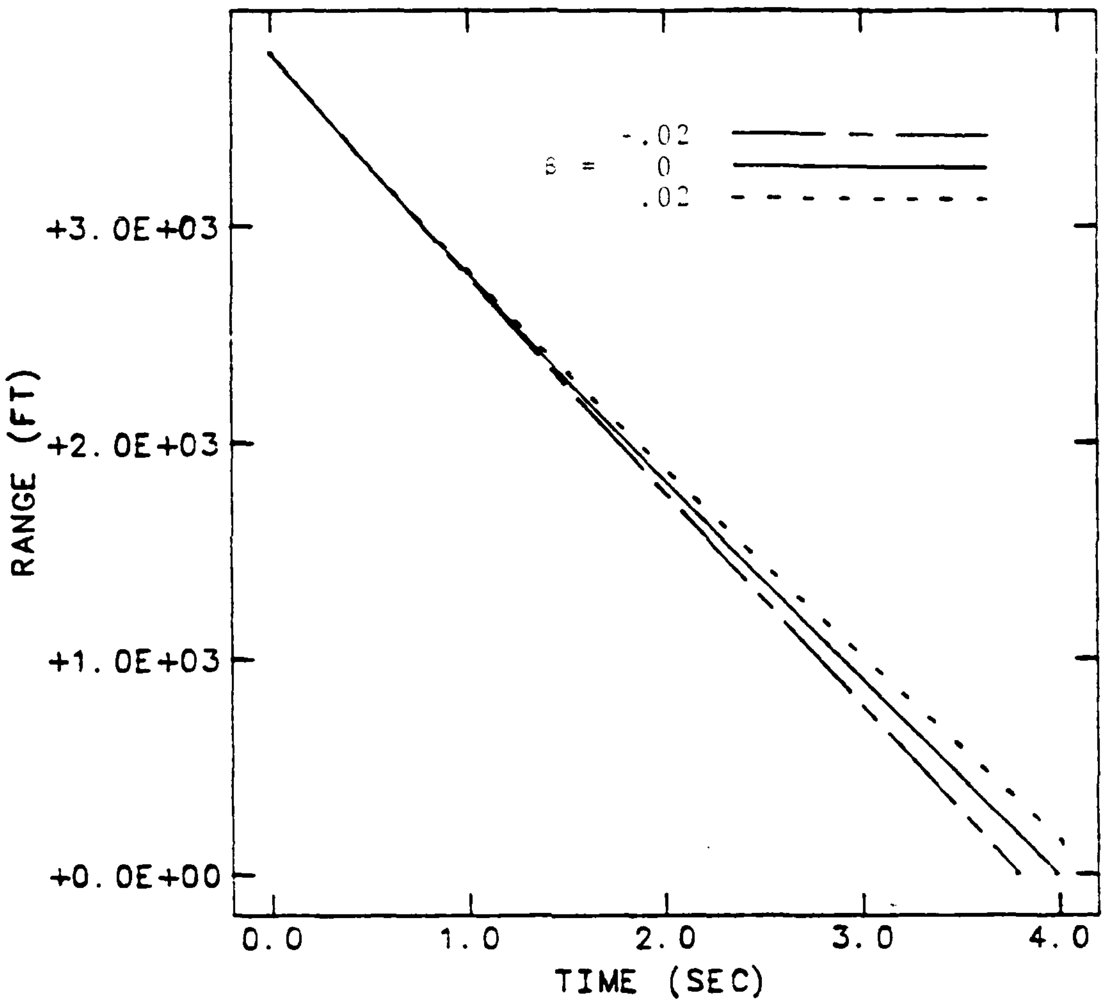


Figure 6.20, Range For Bias Error

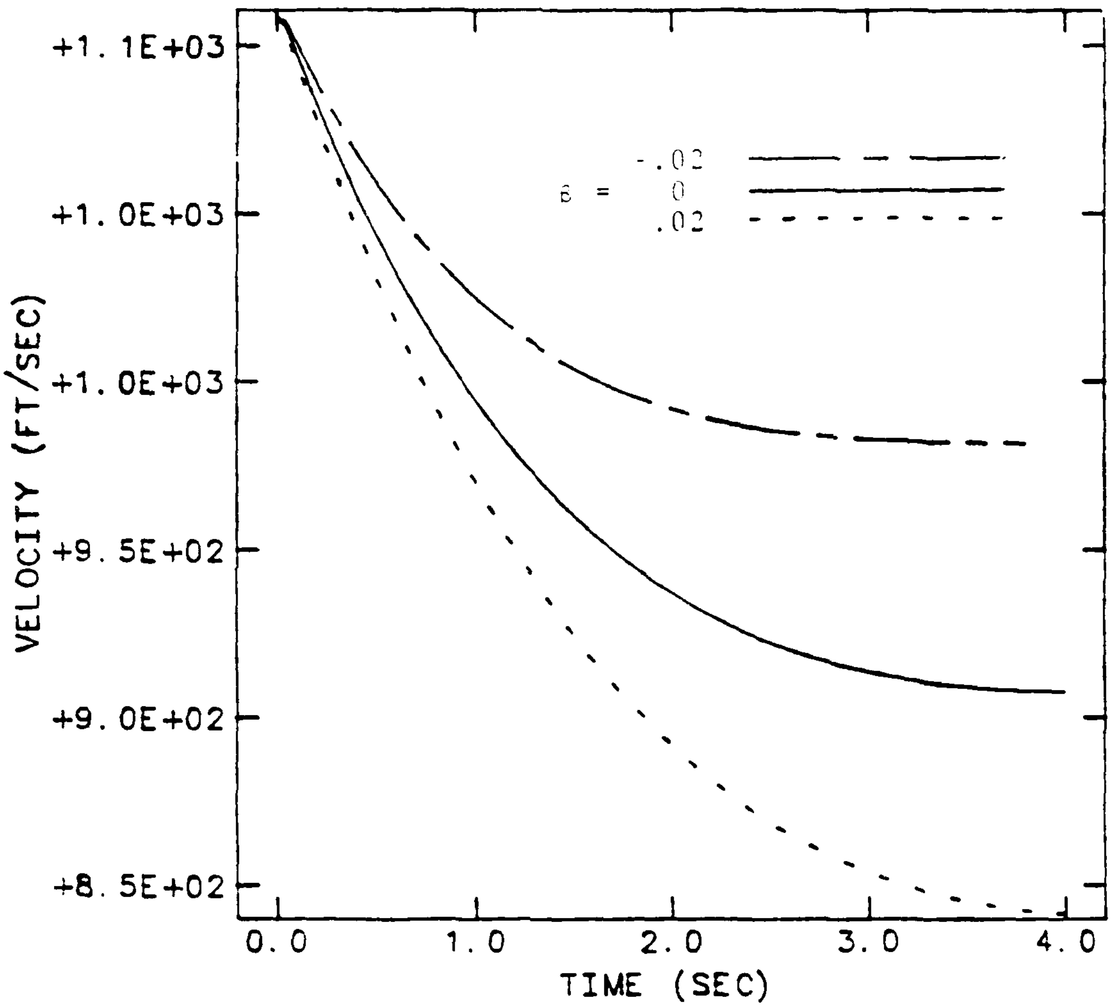


Figure 6.21, Velocity For Bias Error

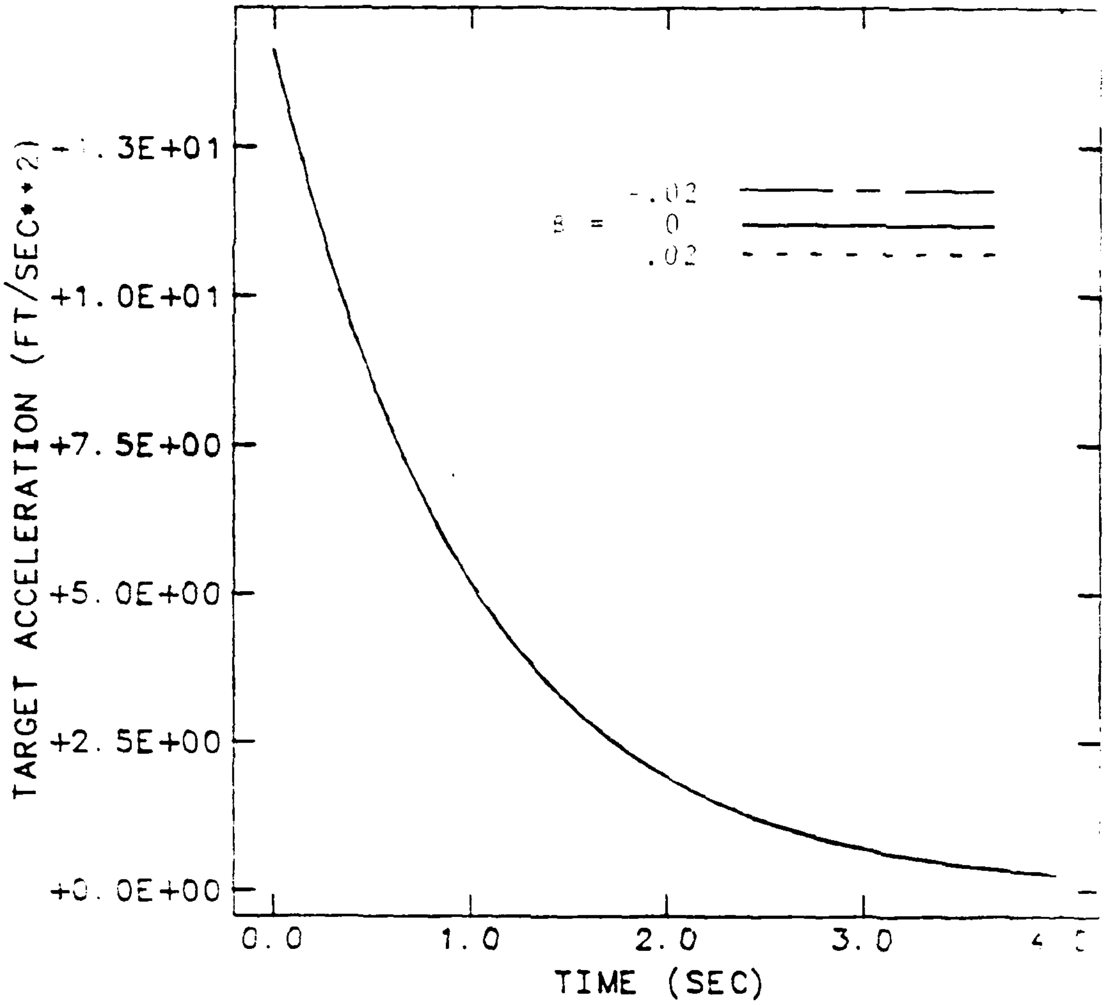


Figure 6.22, Target Acceleration For Bias Error

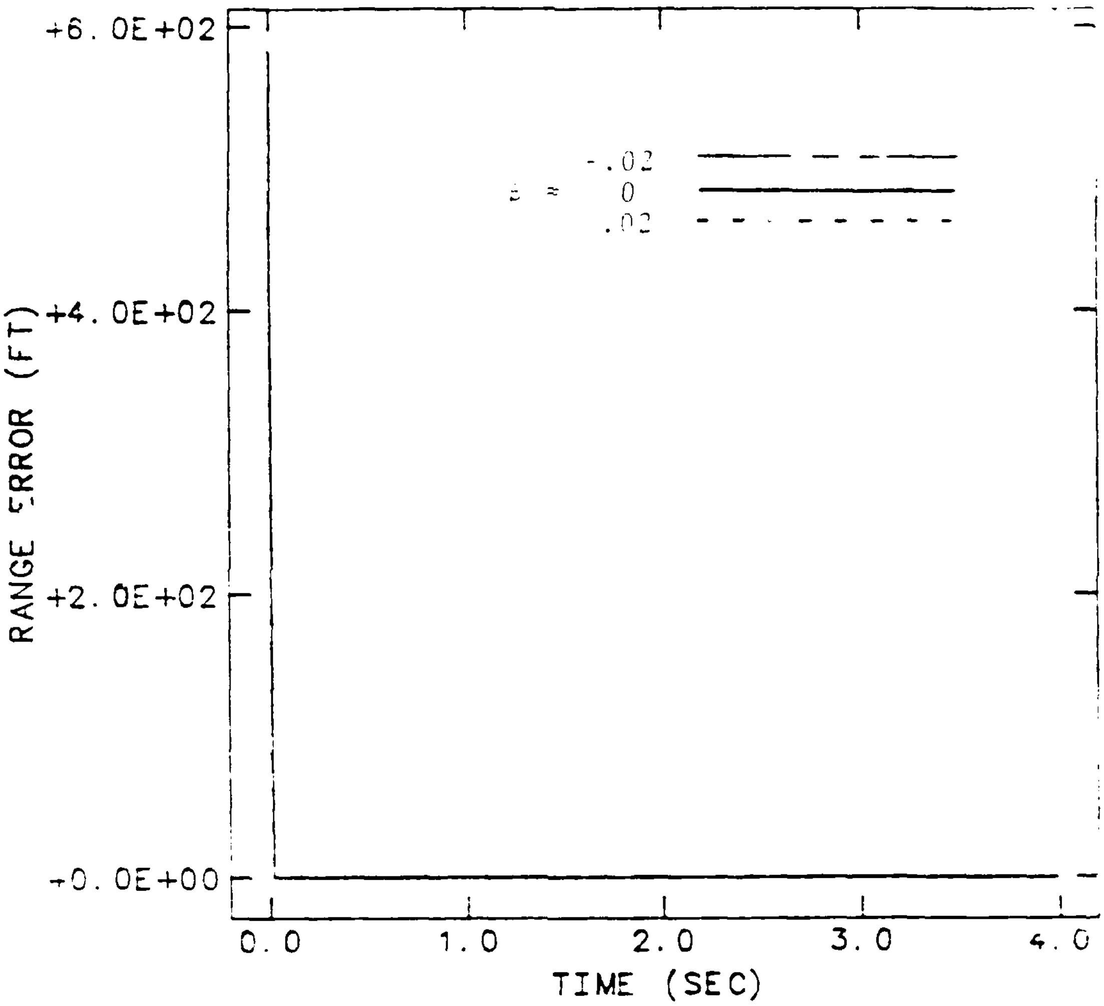


Figure 6.23, Range Error For Bias Error

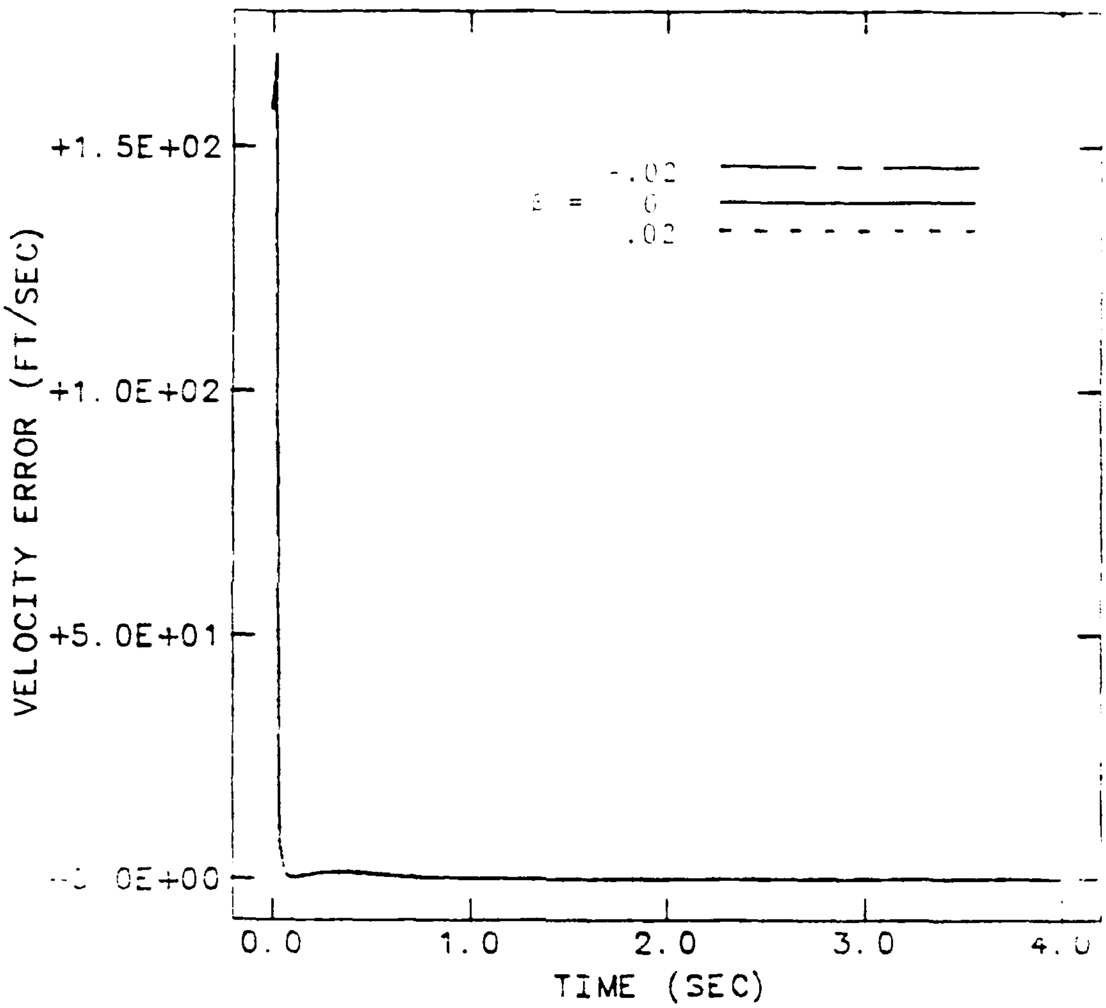


Figure 6.24, Velocity Error For Bias Error

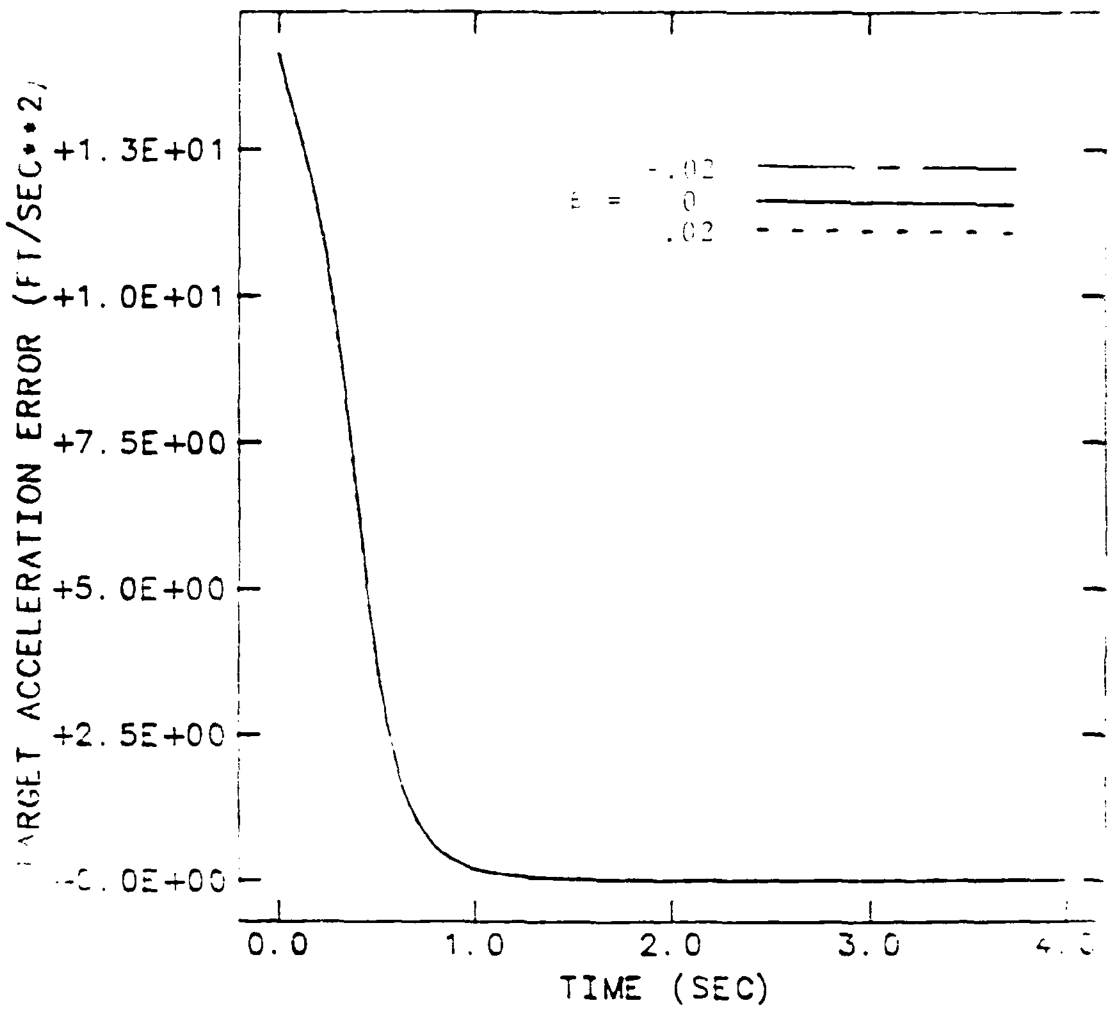


Figure 6.25, Target Acceleration Error For Bias Error

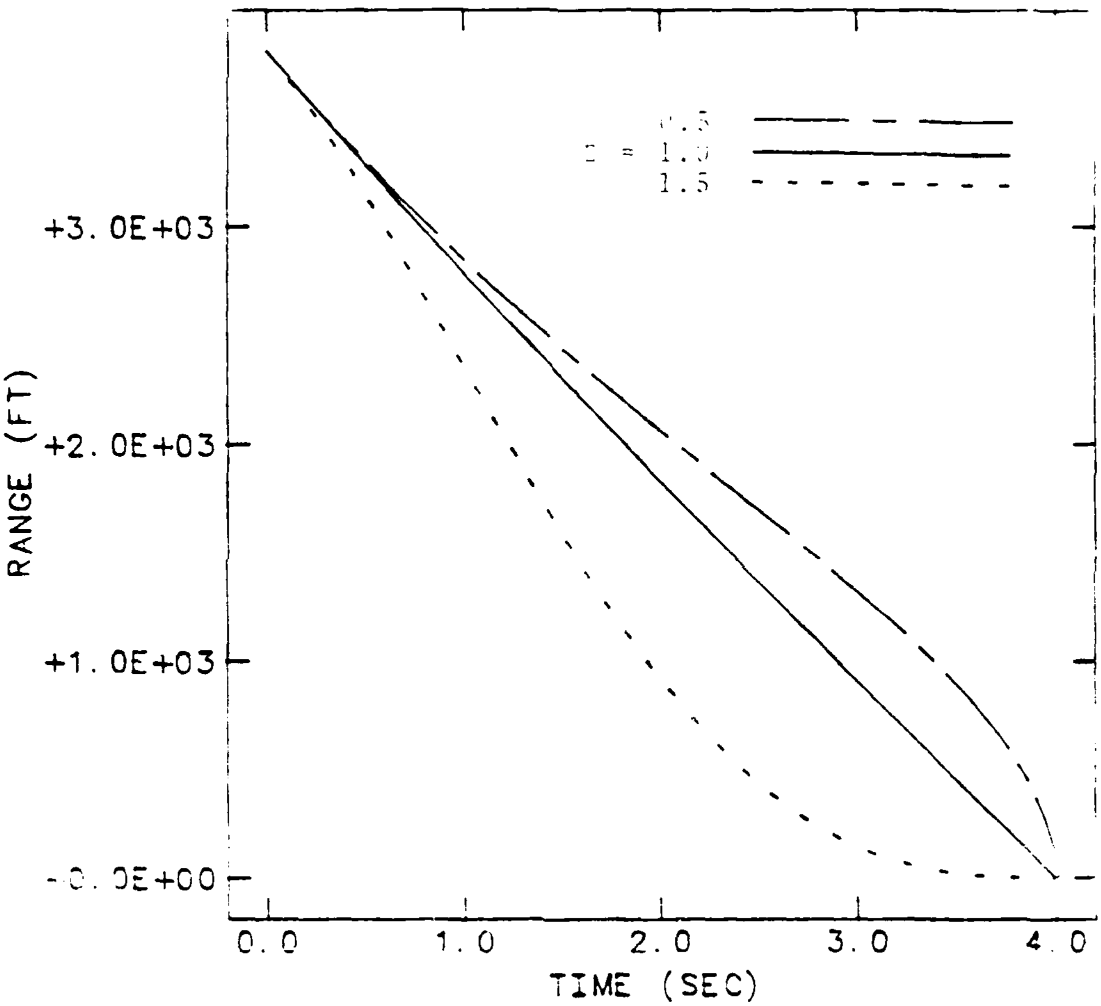


Figure 6.26, Range For Scale Factor Error

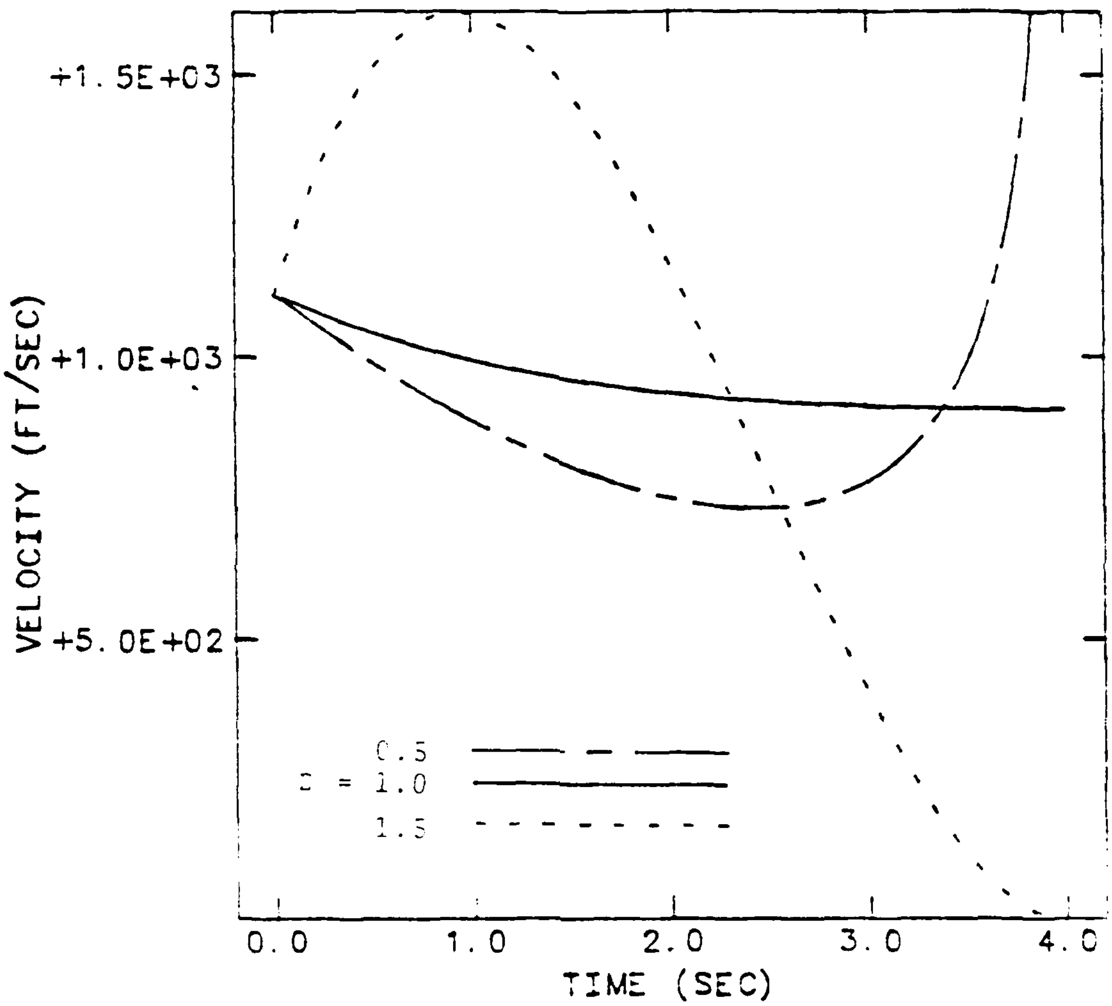


Figure 6.27, Velocity For Scale Factor Error

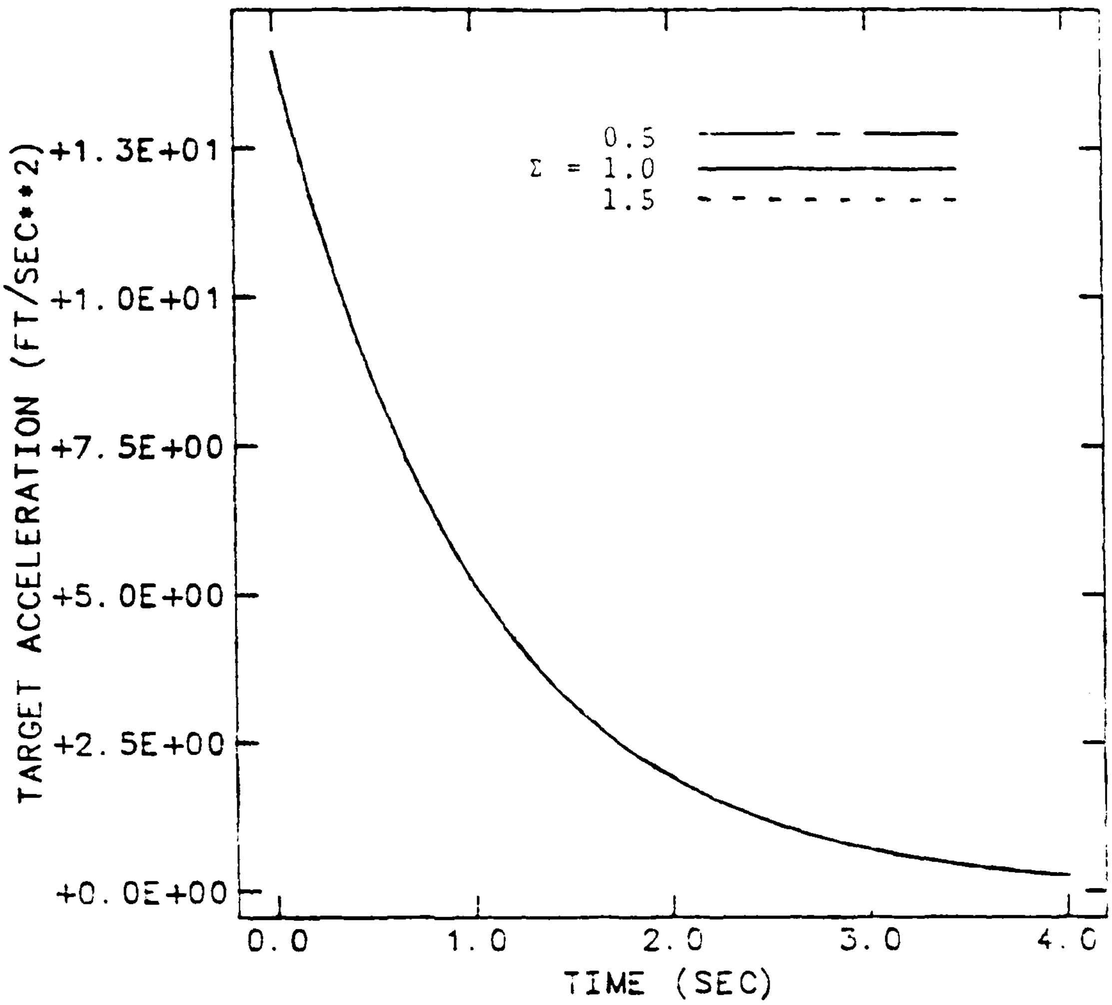


Figure 6.28, Target Acceleration For Scale Factor Error

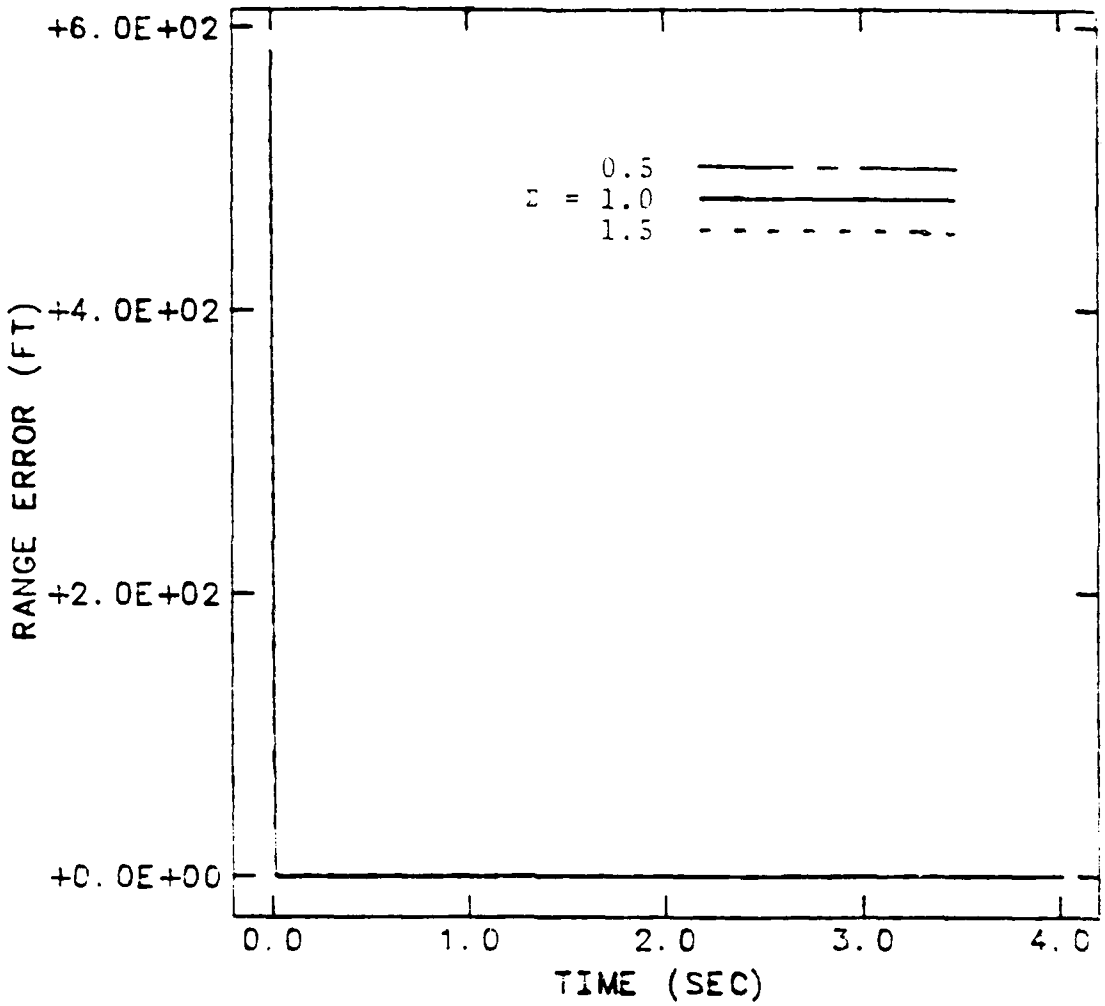


Figure 6.29, Range Error For Scale Factor Error

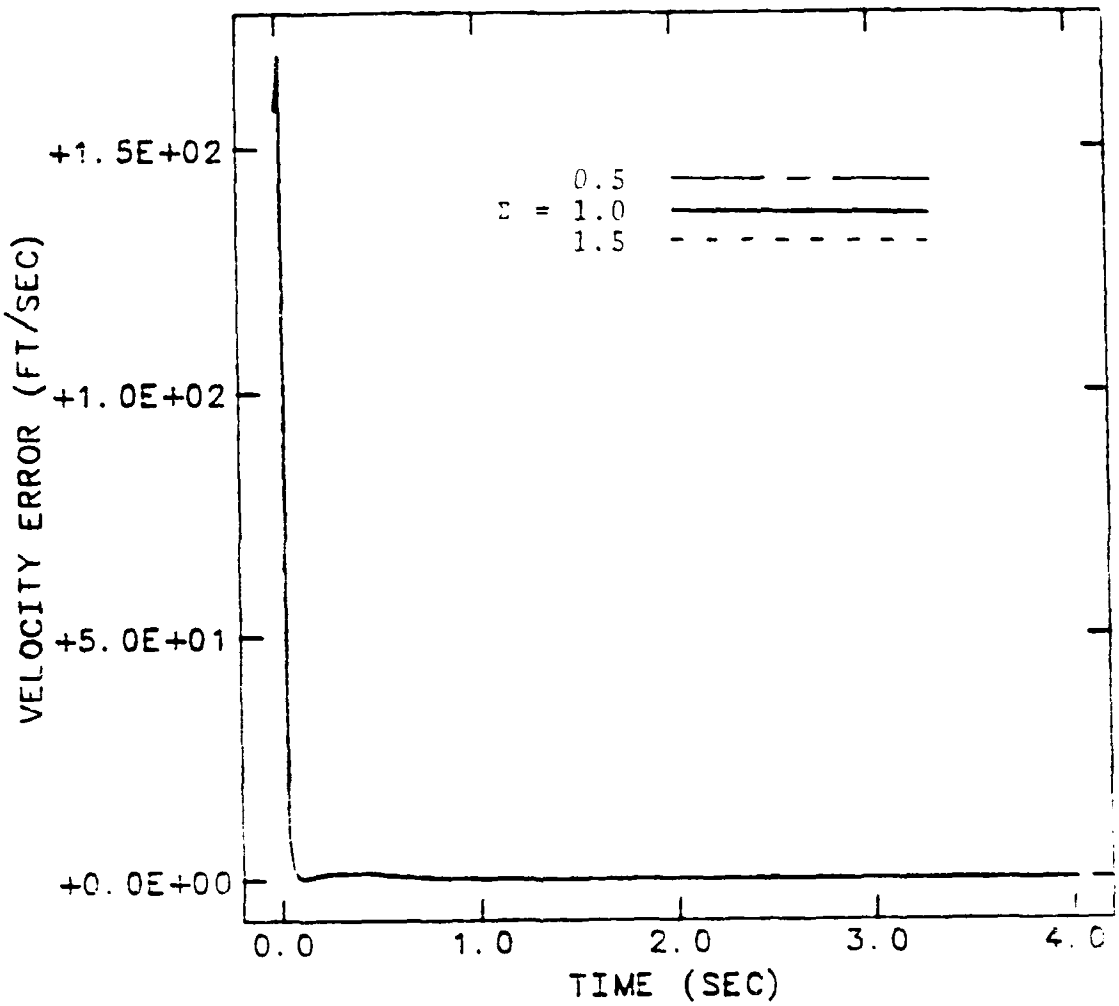


Figure 6.30, Velocity Error For Scale Factor Error

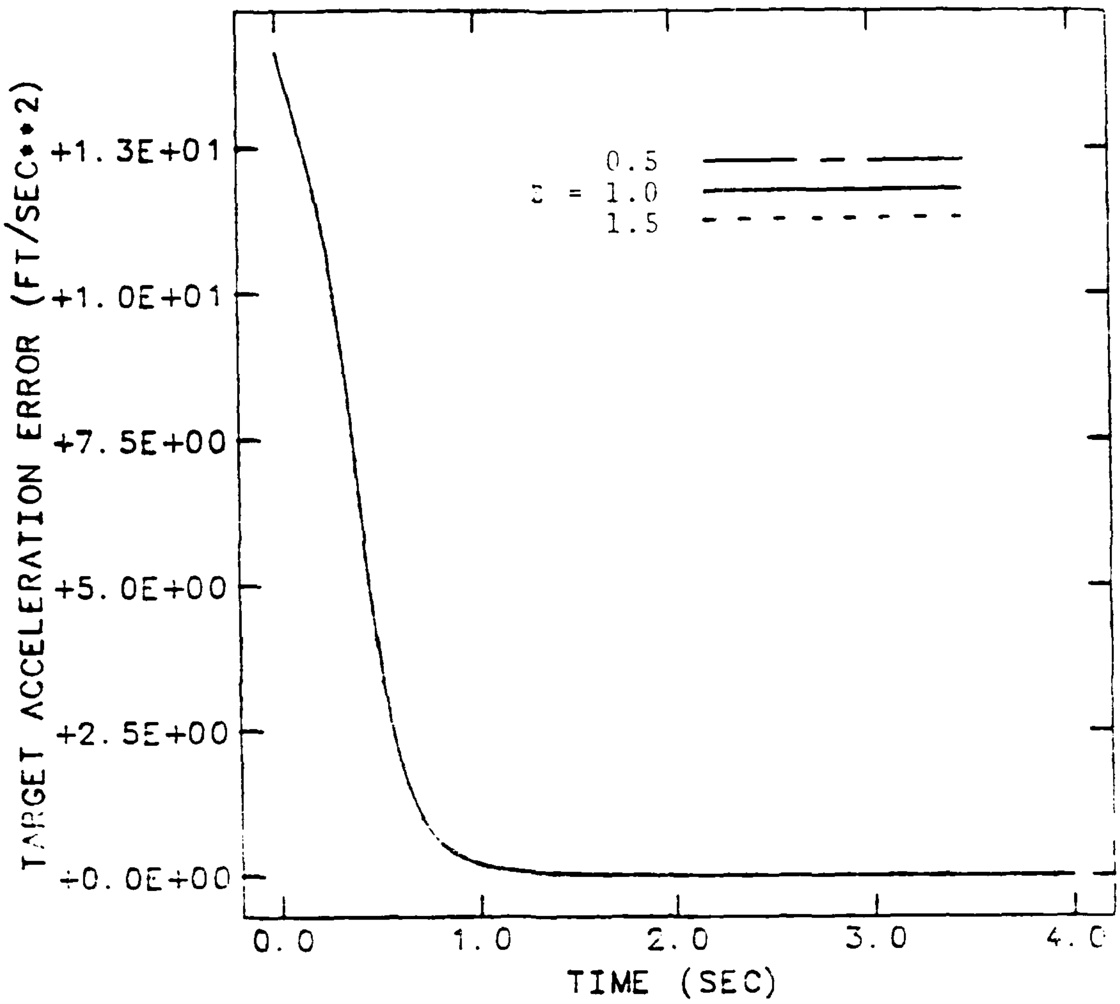


Figure 6.31, Target Acceleration Error For Scale Factor Error

6.3.3 Target Acceleration Modelling Errors

System parameter uncertainties and their effects on system stability are discussed in Section IV. Since the emphasis is on finite time problems, the Lyapunov function derived without parameter uncertainties (equation (4.21)) and the Lyapunov function derived with parameter uncertainties (equation (4.37)) are used to provide a measure of system performance given parameter uncertainties. This section focuses on errors in the system matrix, A; in particular, errors in the target acceleration time constant, λ_T. As in the steady-state analysis of Sections 6.2.1.2 and 6.2.2.1, the first Lyapunov function is valid under a very narrow region around the true value of λ_T. The Lyapunov function derived with parameter variations is valid for a range of variations in λ_T.

Consider the designed system matrix to be defined, as in Section IV, in the following way

$$A_c = A - \Delta A \qquad (6.61)$$

where ΔA is the modelling error and involves errors in λ_T only. Using the same system defined in Section 6.3, simulation runs were generated for errors in A, (ΔA), defined as the following

$$\Delta A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta\bar{A} \end{bmatrix} \qquad (6.60)$$

where $\Delta\bar{A}$ = 0, 1, 2, and 3.

For these values of $\Delta\bar{A}$, the Lyapunov function without parameter uncertainties and the Lyapunov function with parameter uncertainties are evaluated the same way as in the time-to-go error analysis. The combined Lyapunov function will not be used for any further analysis since the same system is used, and it is shown that this Lyapunov function has a positive slope for zero parameter errors.

Figure 6.32 is a plot of the maximum eigenvalue of equation (4.65) for the Lyapunov function derived without parameter uncertainties. For $\Delta\bar{A}=0$, the eigenvalue is approximately -10^{-12} for $0 \leq t < t_f$. This small a number cannot be seen on the figure. For $\Delta\bar{A}=10^{-7}$, the maximum eigenvalue becomes positive, thus invalidating this Lyapunov function. Figure 6.33 shows the minimum eigenvalue of equation (4.82) for the Lyapunov equation derived with parameter uncertainties. Figures 6.34 and 6.35 show the maximum eigenvalue for the same Lyapunov equation. This Lyapunov equation indicates good performance for $\Delta\bar{A}=0$ and 1; however, the function becomes unbounded from above for $\Delta\bar{A}$ =2 and 3. Thus, for $\Delta\bar{A} \geq 2$, this Lyapunov equation indicates that the system will not perform well for all x and e.

Figures 6.36-6.43 are plots of the magnitude of relative position, velocity, and target acceleration; as well as their errors from the estimation algorithm for the set of launch conditions specified in Section 6.3. These results are useful in showing that the Lyapunov function without parameter uncertainties is a poor measure of performance for this system with target acceleration modelling errors.

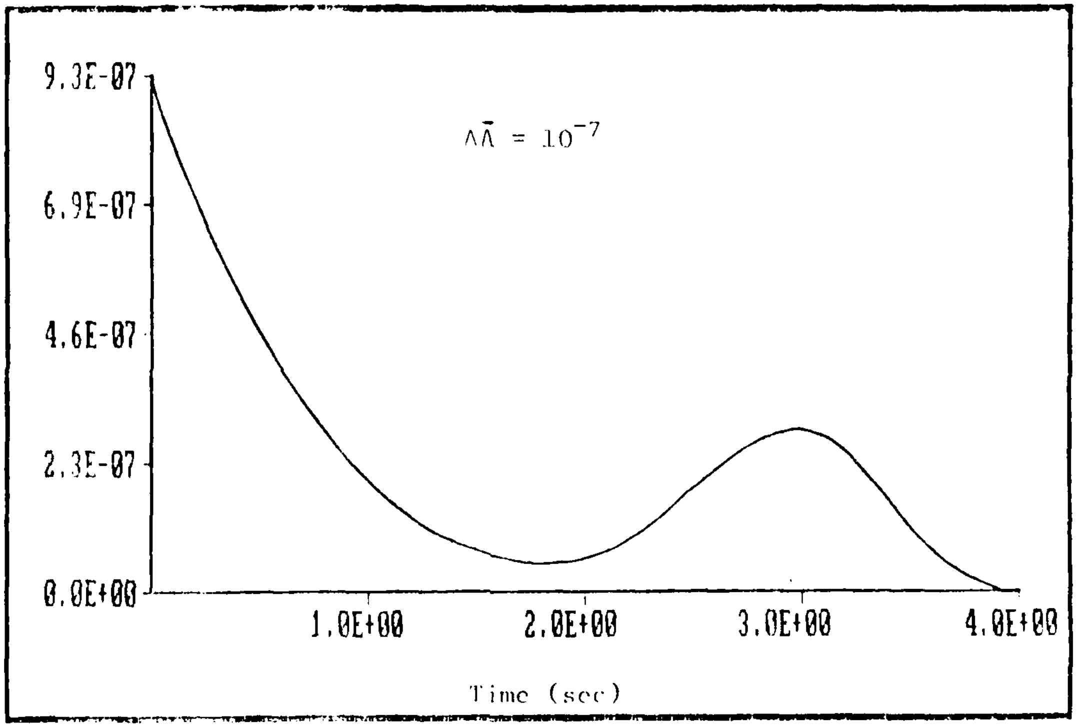


Figure 6.32, Maximum eigenvalue of equation (4.65) for the Lyapunov function derived without parameter uncertainties

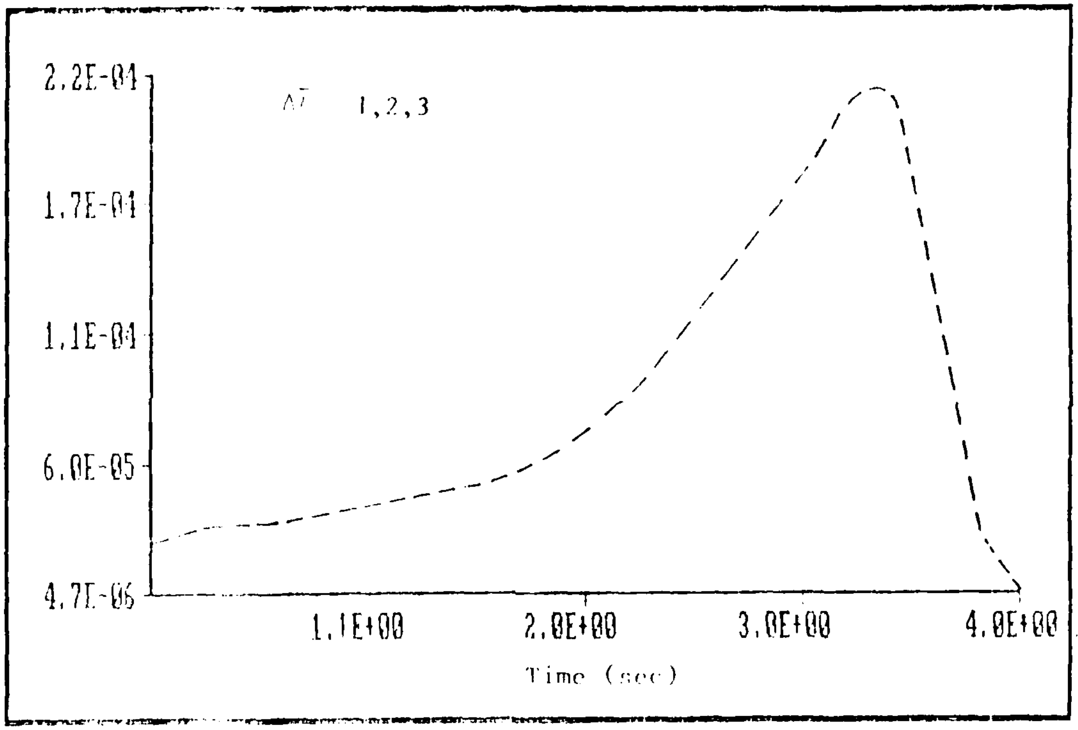


Figure 6.33. Minimum eigenvalue of the Lyapunov equation for the Lyapunov function derived with parameter uncertainties

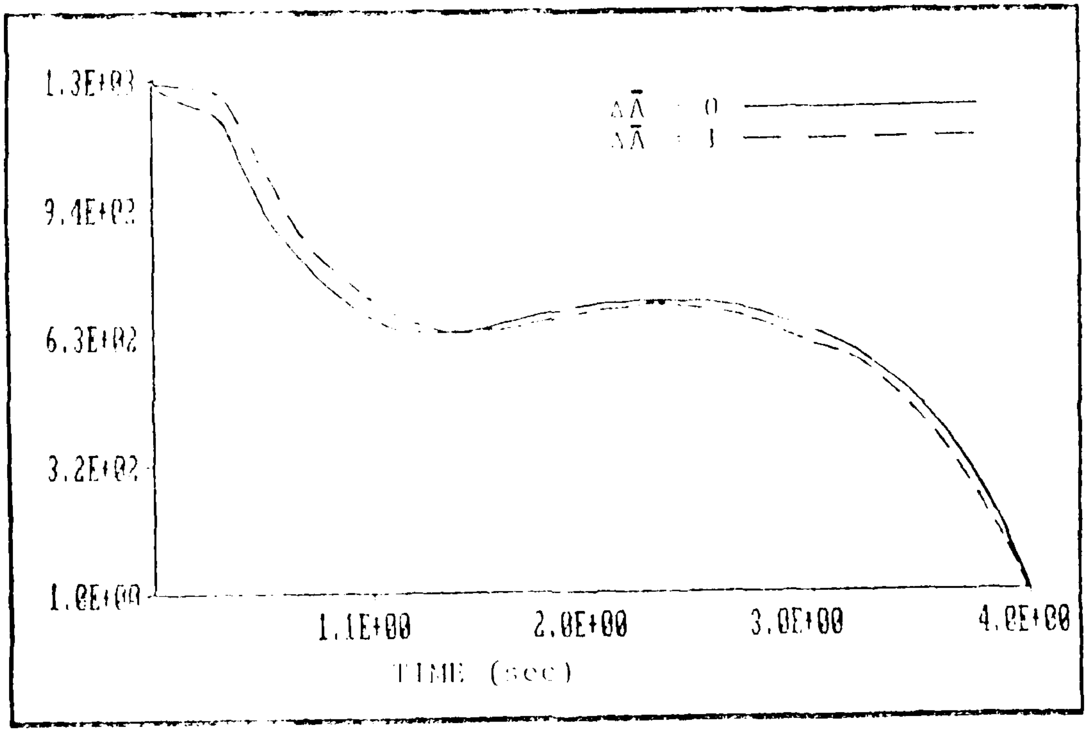


Figure 6.34, Maximum Eigenvalue of the Lyapunov Equation for the Lyapunov Function Derived with Parameter Uncertainties

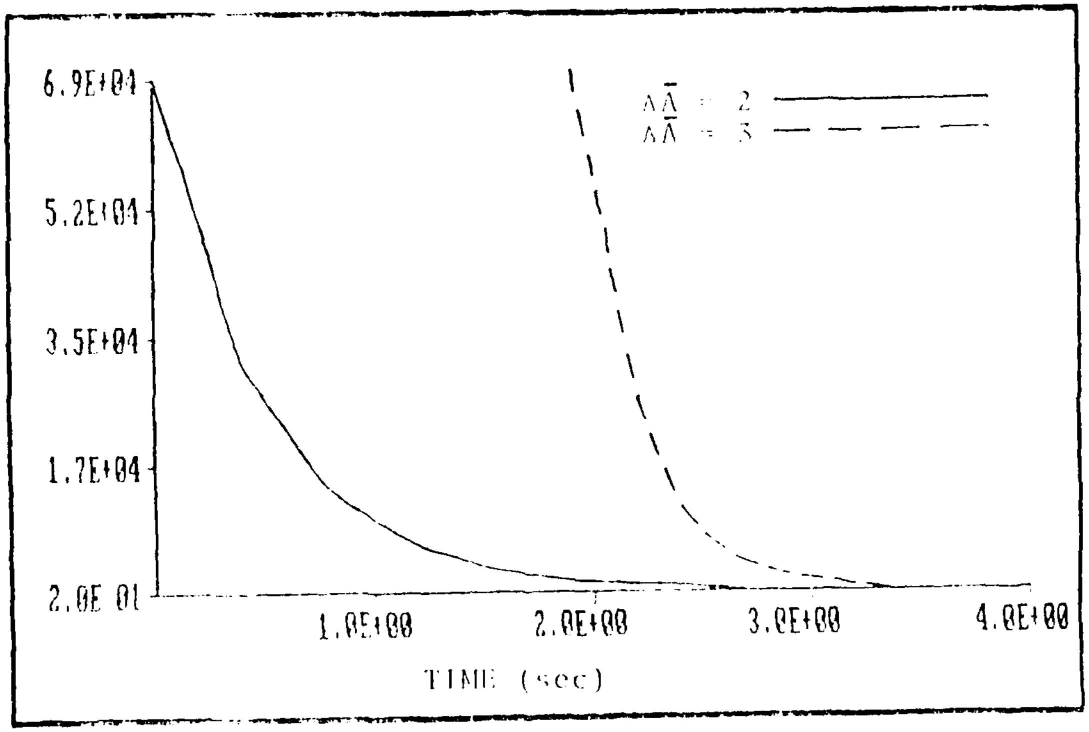


Figure 6.35, Maximum Eigenvalue of the Lyapunov Equation for the Lyapunov Function Derived with Parameter Uncertainties

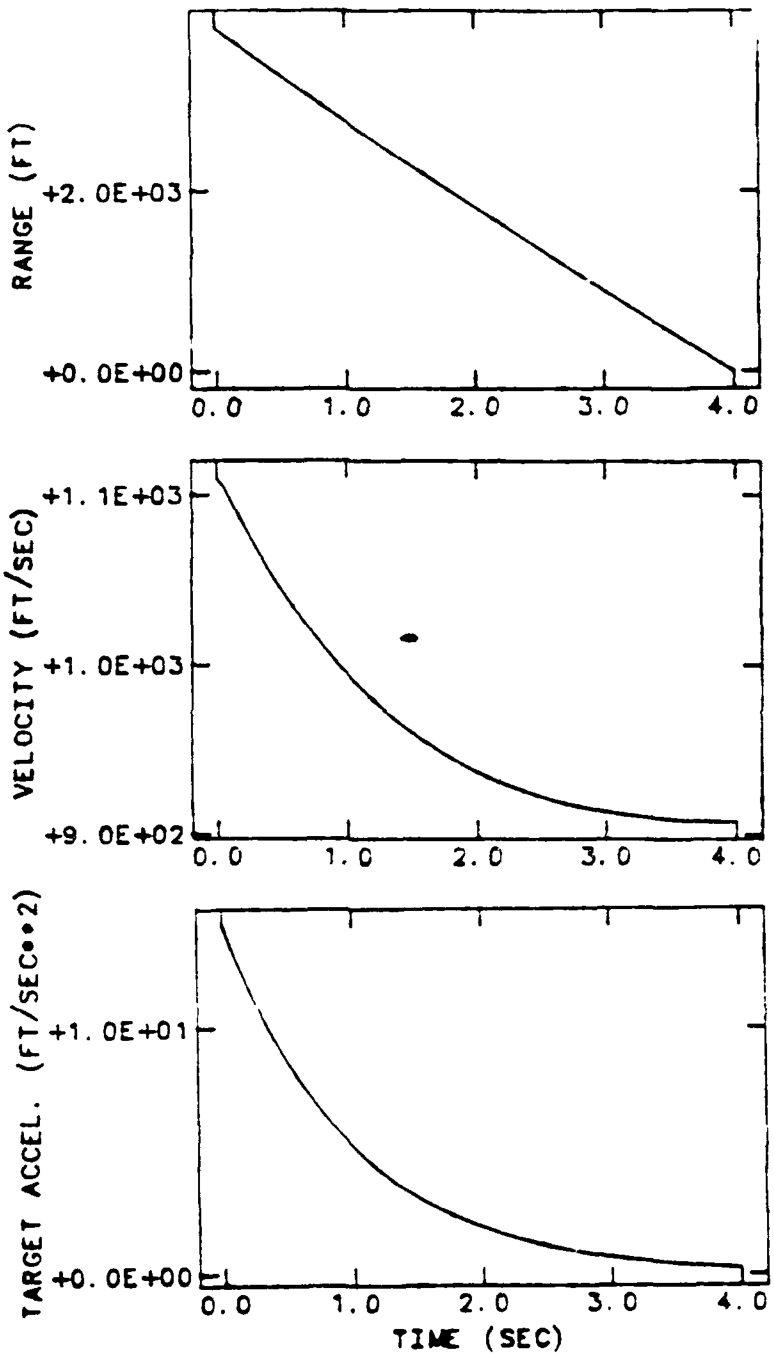


Figure 6.36, Range, Velocity, and Target Acceleration
$\Delta\bar{A} = 0$

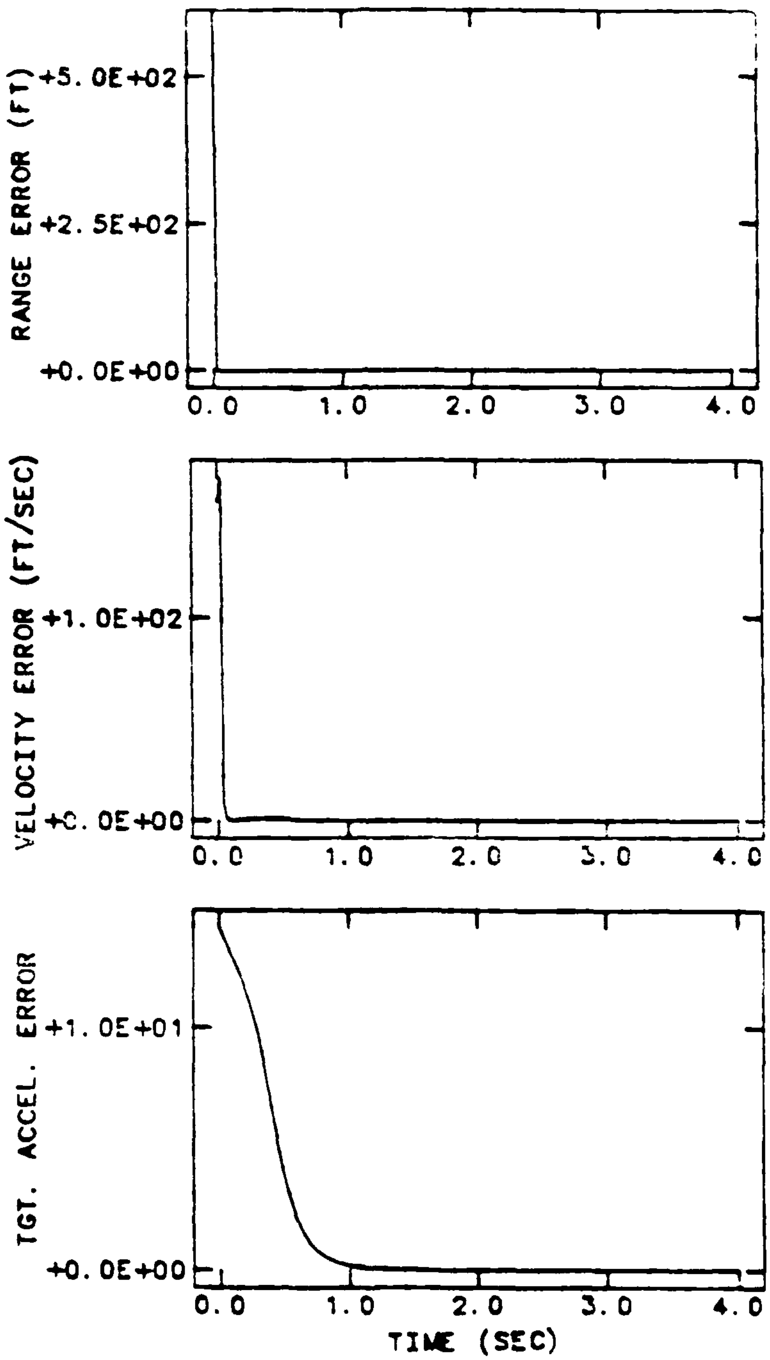


Figure 6.37, Range, Velocity, and Target Acceleration Errors $\Delta\bar{A} = 0$

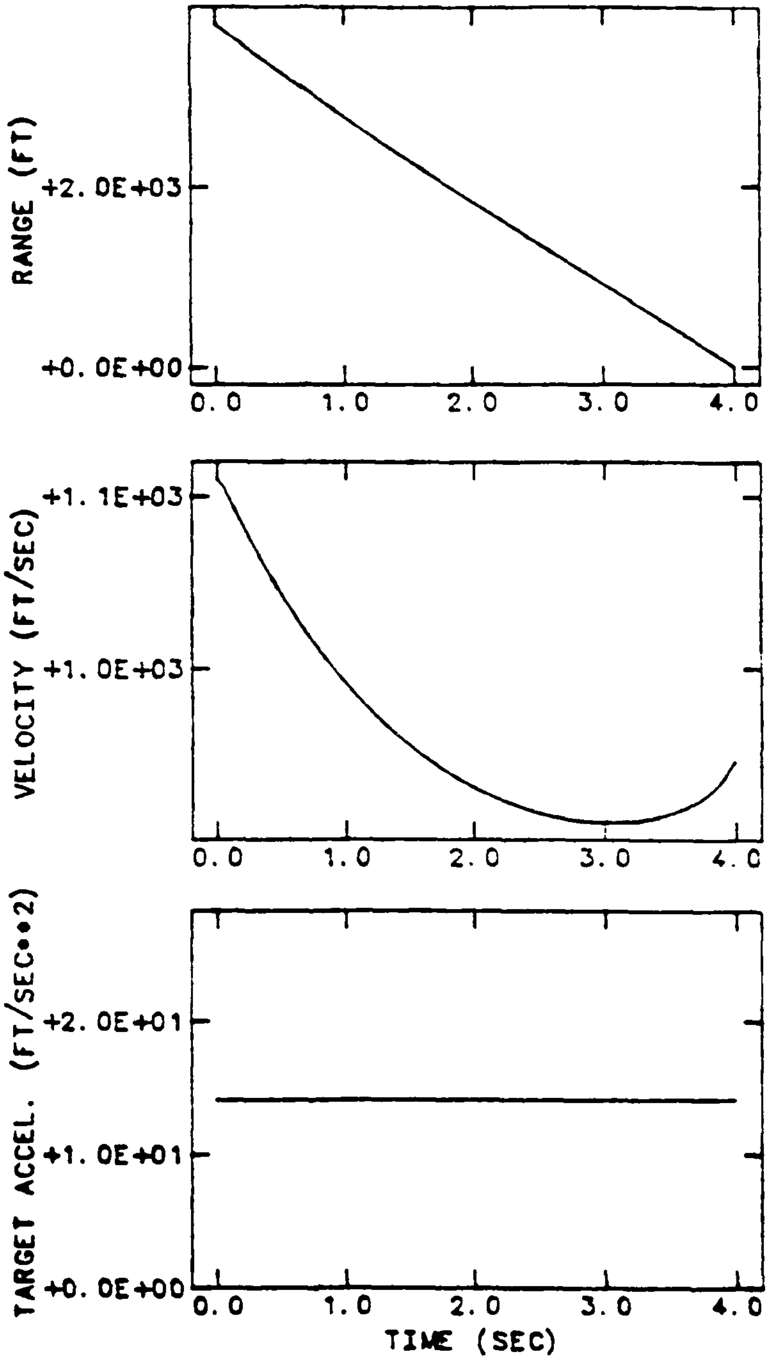


Figure 6.38, Range, Velocity, and Target Acceleration
$\angle A = 1$

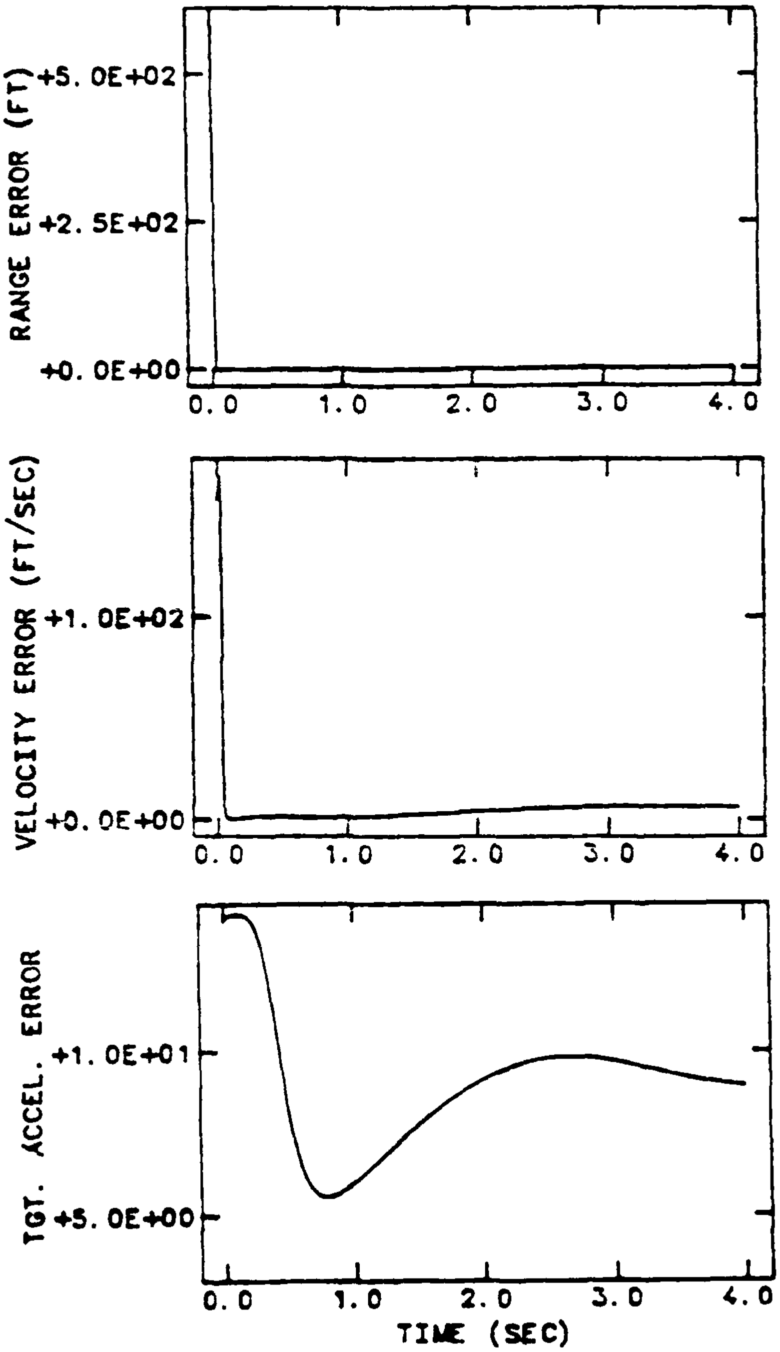


Figure 6.39, Range, Velocity, and Target Acceleration Errors $\underline{\Delta A} = 1$

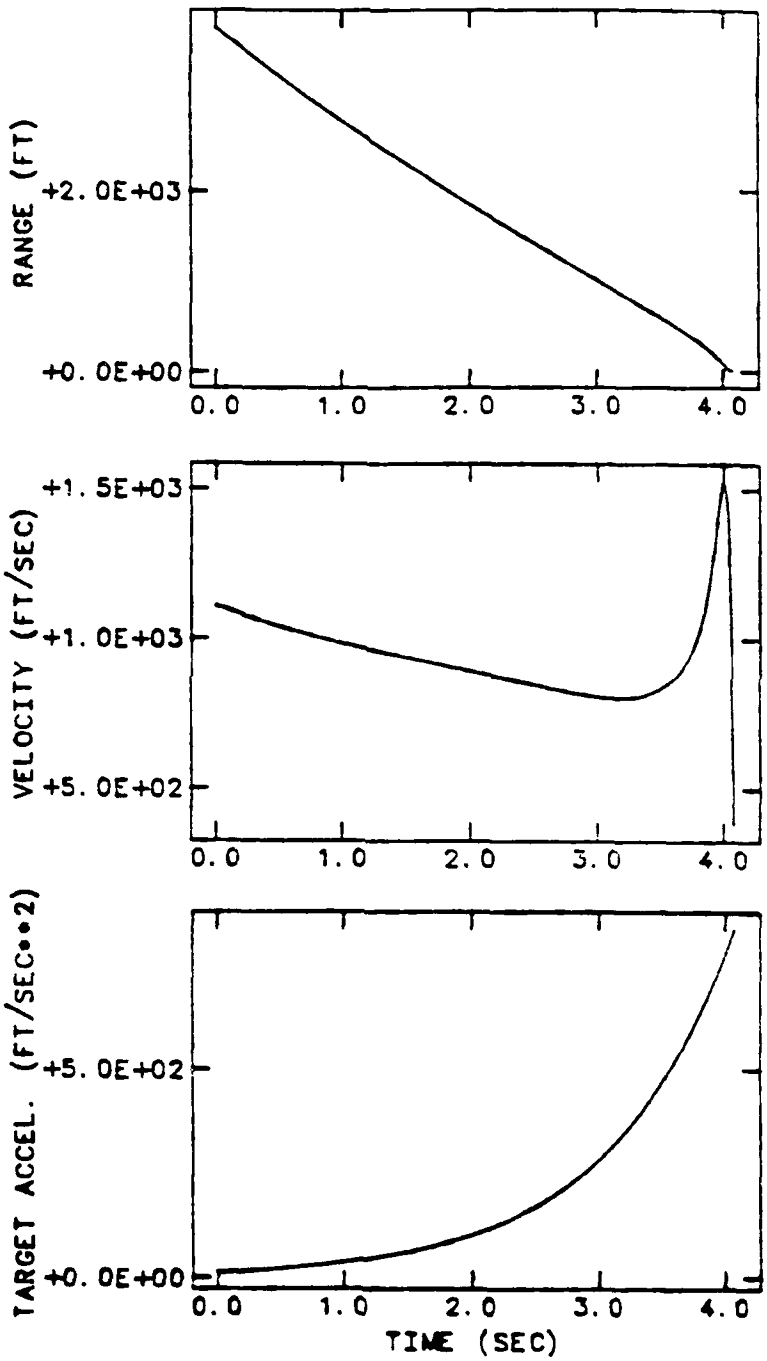


Figure 6.40, Range, Velocity, and Target Acceleration
$\Delta\bar{A} = 2$

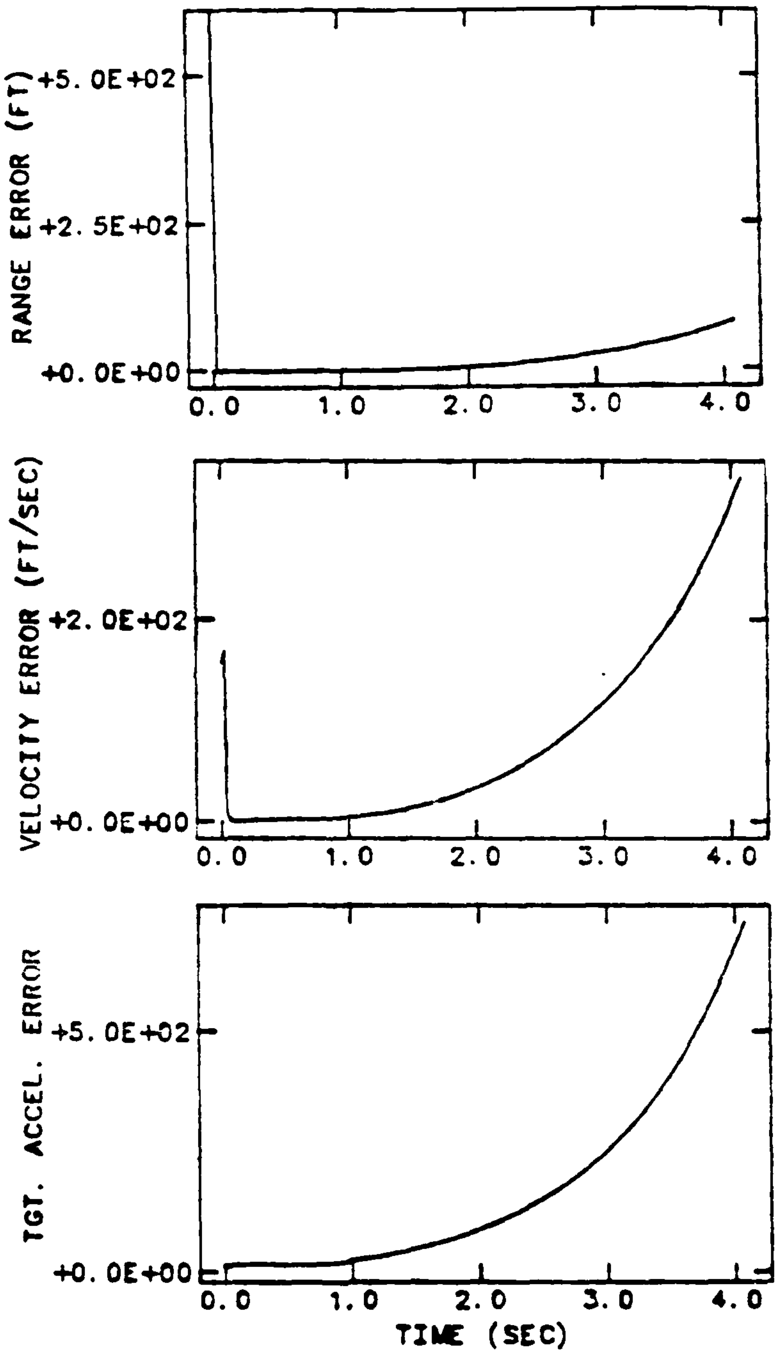


Figure 6.41, Range, Velocity, and Target Acceleration Errors $\Delta\bar{A}$ = 2

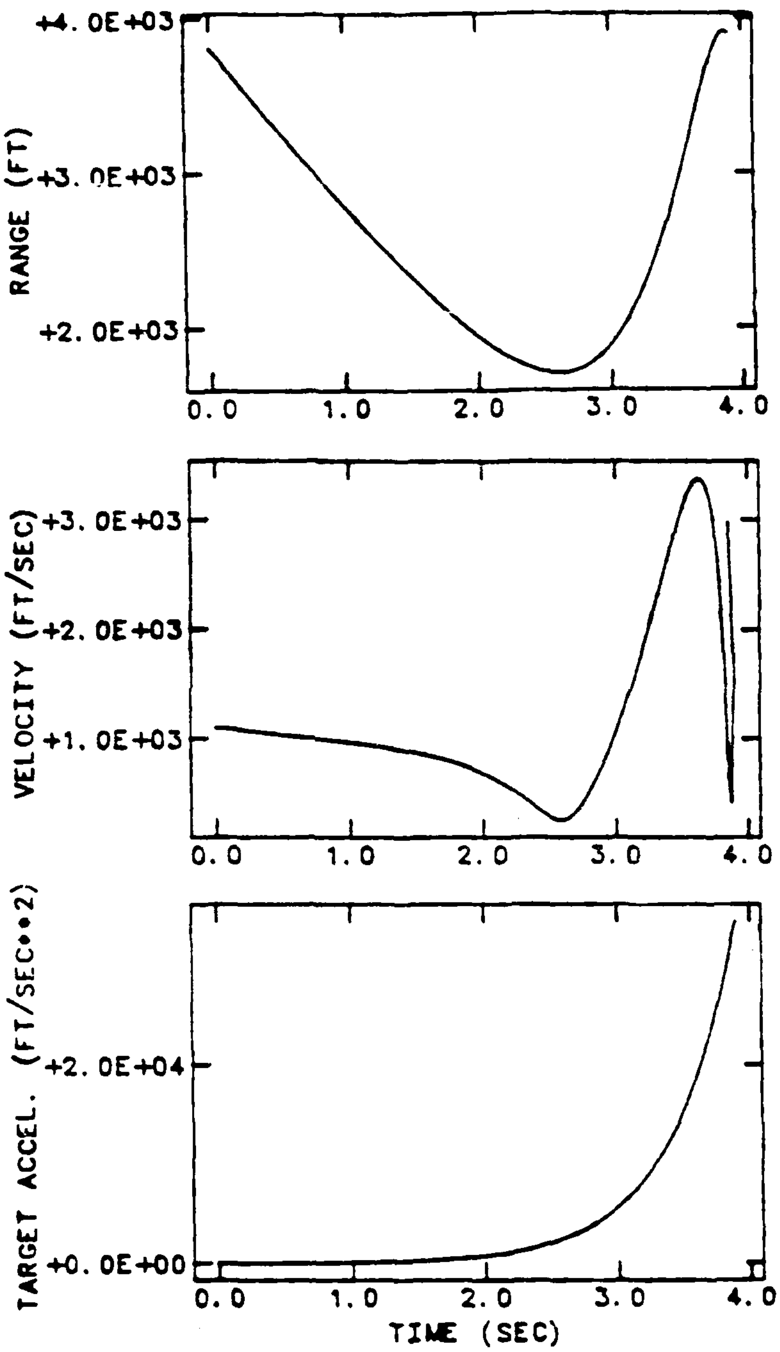


Figure 6.42, Range, Velocity, and Target Acceleration
$\Delta\bar{A} = 3$

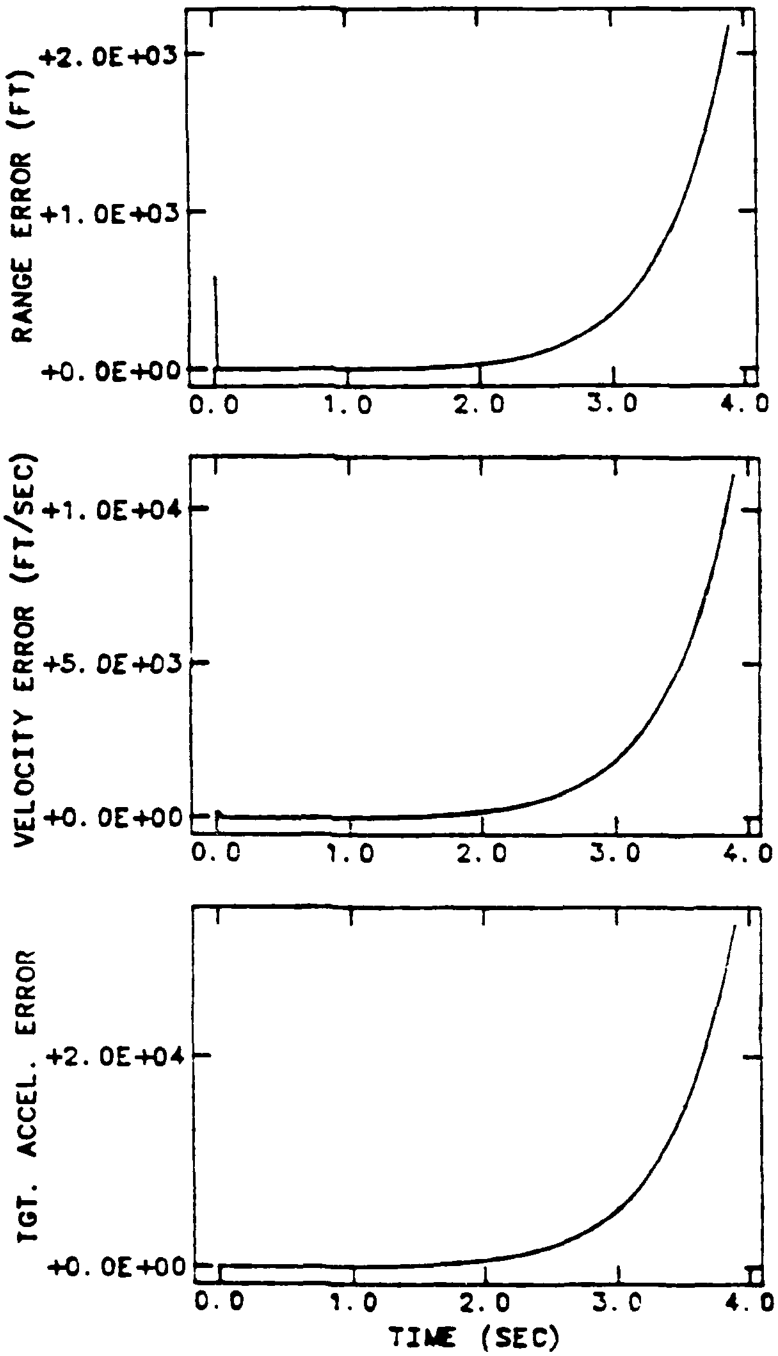


Figure 6.43, Range, Velocity, and Target Acceleration Errors $\angle\overline{A}$ = 3

6.4 Homing Missile Guidance Problem with Angle Only Measurements

The system selected for this analysis is identical to the one defined in Section 6.3, except that the measurement model is a nonlinear function of the system states representing an angular measurement from the following figure.

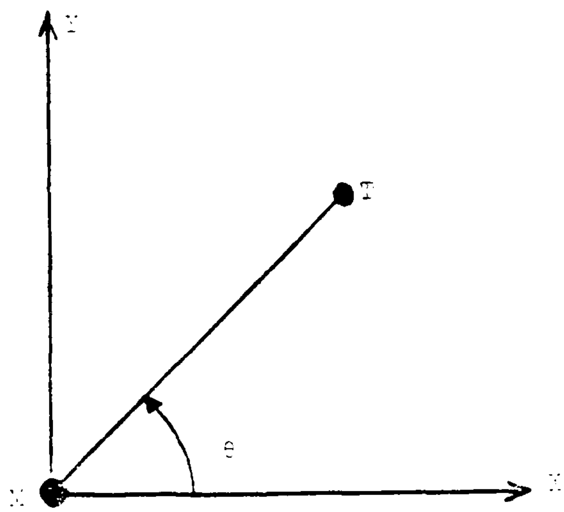


Figure 6.44, Angular Measurement Related to System States

The measurement model now becomes

$$y = h(x) + \nu \quad , \quad \nu \sim N(0, R_o) \tag{6.62}$$

where $R_o = .1$ and

$$\Theta = h(x) = \tan^{-1}\left(\frac{Y}{X}\right) \tag{6.63}$$

The estimation algorithm can no longer be a linear Kalman observer, as in Section 6.3, since the measurement model is nonlinear.

6.4.1 Pseudomeasurement Observer

The pseudomeasurement observer (PMO) is selected as the estimation algorithm because it is reasonably easy to mechanize (like the extended Kalman observer) and it has global convergence properties [119,120]. The algorithm for the PMO is [119,120]

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y^{*} - h(\hat{x})) \quad , \quad \hat{x}(t_{\emptyset}) = \hat{x}_{\emptyset} \tag{6.64}$$

where

$$K = pg(y^{*},\hat{x})R_{o}^{-1} \tag{6.65}$$

$$\dot{p} = Ap + pA^{T} + Q_{o} - ph^{T}(\hat{x})R_{o}^{-1}h(\hat{x})p$$

$$- pg^{T}R_{o}^{-1}h(\hat{x})p + ph^{T}(\hat{x})R_{o}^{-1}h(\hat{x})p \tag{6.66}$$

$$y^{*} = h(x) \tag{6.67}$$

The definition for modifiable [119,120] is that a time-varying function $h: R^n \to R^p$ is a modifiable non-linear system function if there exists a p x n time-varying matrix of functions $g: R^q \times R^n \to R^{p \times n}$ so that for any $x, \bar{x} \in R^n$ and $y \in R^q$,

$$h(x) - h(\bar{x}) = g(y^*, \bar{x})(x - \bar{x}) \qquad (6.68)$$

and

$$h(x) - h(\hat{x}) = g(y^*, \hat{x})(x - \hat{x}) \qquad (6.69)$$

6.4.2 Target Acceleration Modelling Errors

The target acceleration modelling error analysis follows the same work discussed in Section 6.3.3, where the target acceleration modelling error comes from equations (6.60) and (6.61), and where $\Delta\bar{A}$ is selected as 0, 1, 2, and 3.

For these values of $\Delta\bar{A}$, both the Lyapunov function without parameter uncertainties and the Lyapunov function with parameter uncertainties are evaluated using the PMO algorithm and the initial conditions from Section 6.3 and 6.4. Figure 6.45 is the maximum eigenvalue of equation (4.65) for the Lyapunov function derived without parameter uncertainties. The results indicate that for very small $\Delta\bar{A}$ ($\Delta\bar{A}=10^{-7}$), the maximum eigenvalue becomes positive, thus invalidating this

Lyapunov function. Figures 6.46 and 6.47 show the minimum eigenvalue of equation (4.82) for the Lyapunov function with parameter uncertainties. Figures 6.48 and 6.49 show the maximum eigenvalue for the same Lyapunov equation. This Lyapunov equation shows that the system performs well for $\Delta\bar{A}=0$ and 1. For $\Delta\bar{A}=2$ and 3, the Lyapunov function indicates that the system will not perform well for all x and e.

Figures 6.50-6.57 are plots of the magnitude of relative position, velocity, and target acceleration; as well as their errors from the PMO for the set of launch conditions specified in Section 6.3. Again, these results are useful in showing that the Lyapunov function derived without parameter uncertainties is a poor measure of performance for this system with target acceleration modelling errors.

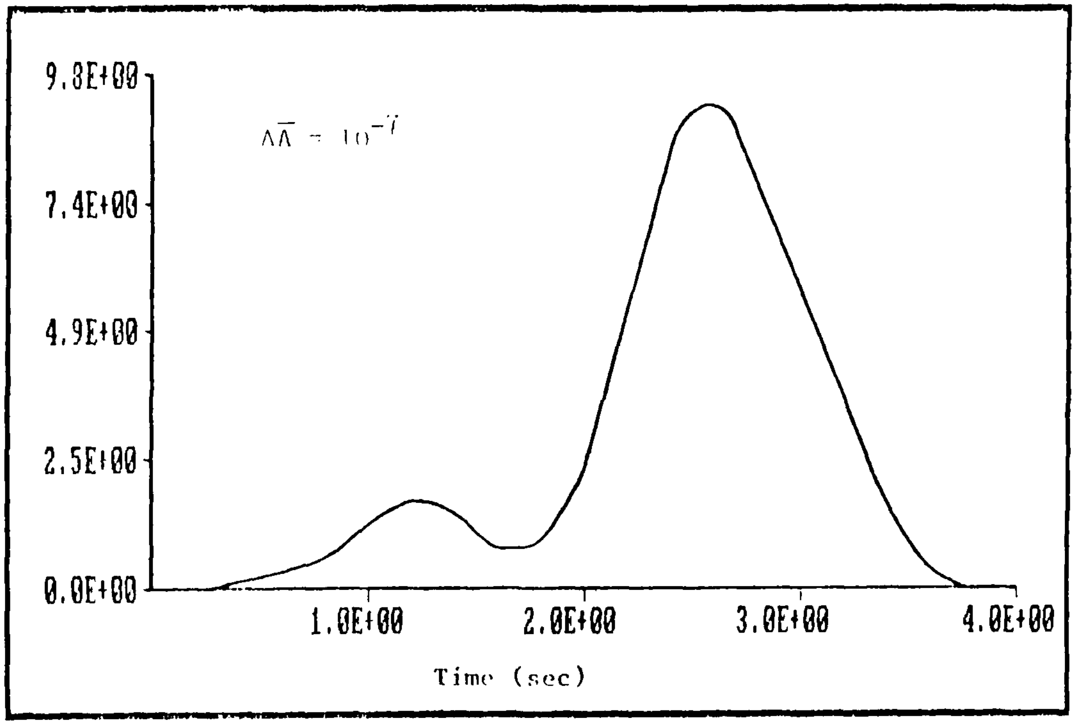


Figure 6.45, Maximum eigenvalue of equation (4.65) for the Lyapunov function derived without parameter uncertainties

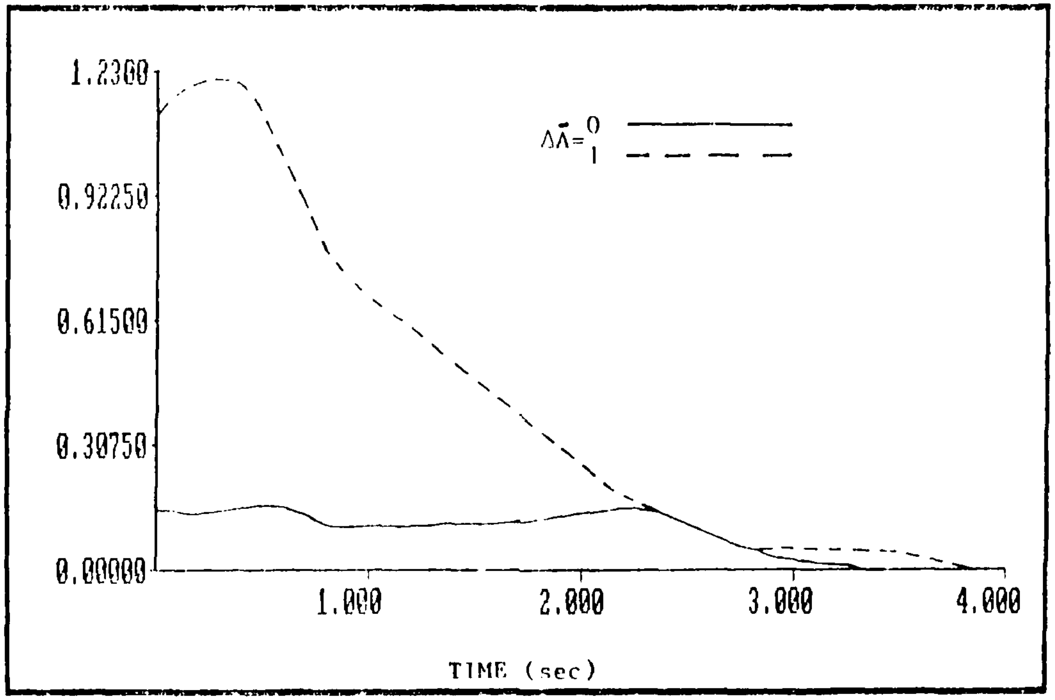


Figure 6.46, Minimum Eigenvalue of the Lyapunov Equation for the Lyapunov Function Derived with Parameter Uncertainties

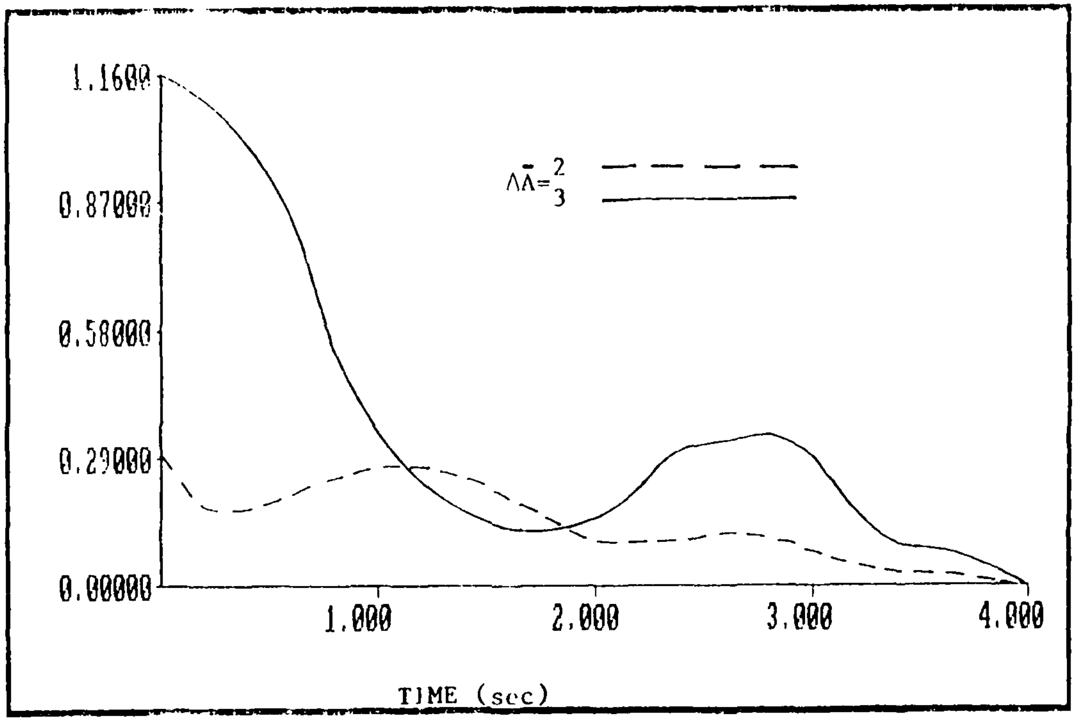


Figure 6.47, Minimum Eigenvalue of the Lyapunov Equation for the Lyapunov Function Derived with Parameter Uncertainties

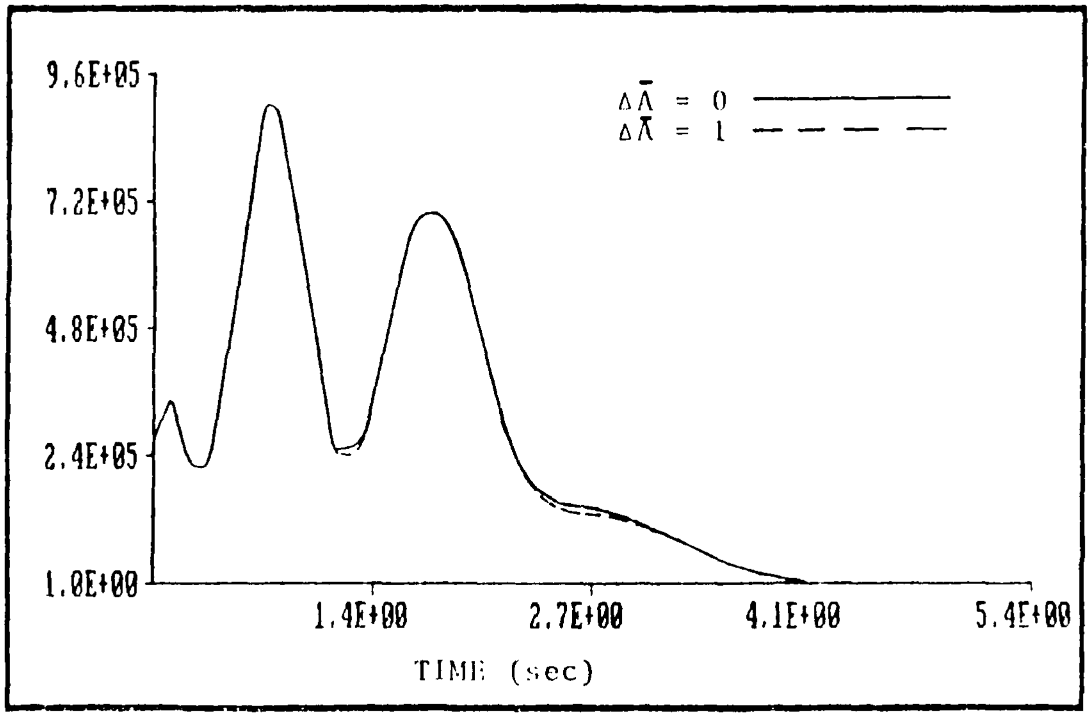


Figure 6.48, Maximum Eigenvalue of the Lyapunov Equation for the Lyapunov Function Derived with Parameter Uncertainties

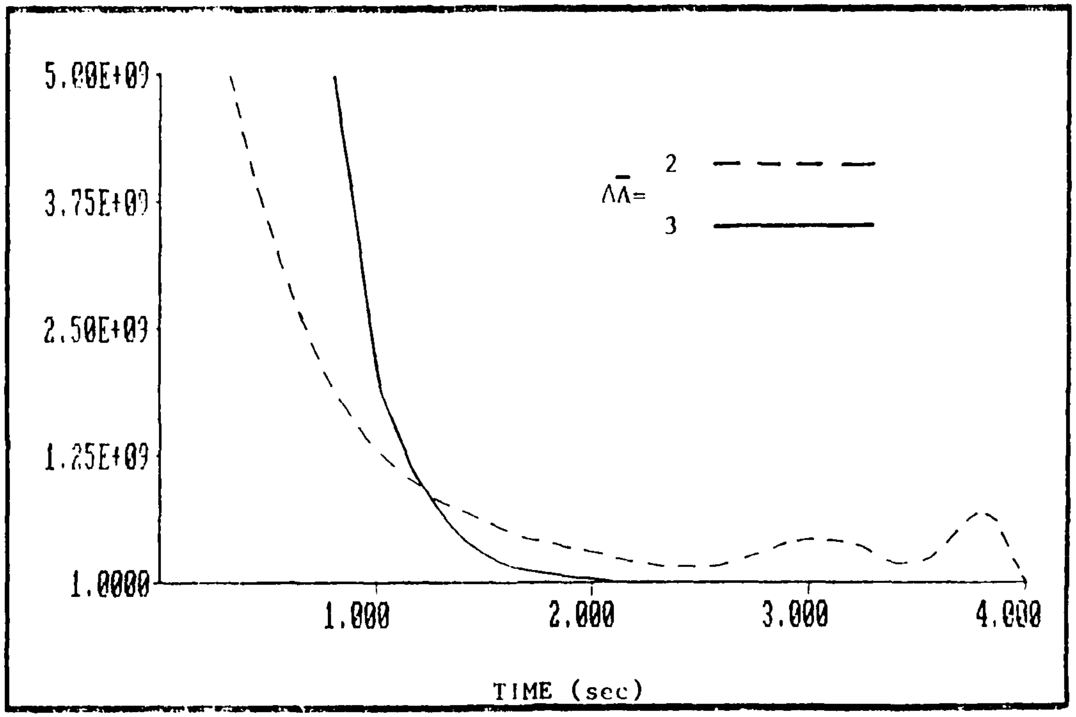


Figure 6.49, Maximum Eigenvalue of the Lyapunov Equation for the Lyapunov Function Derived with Parameter Uncertainties

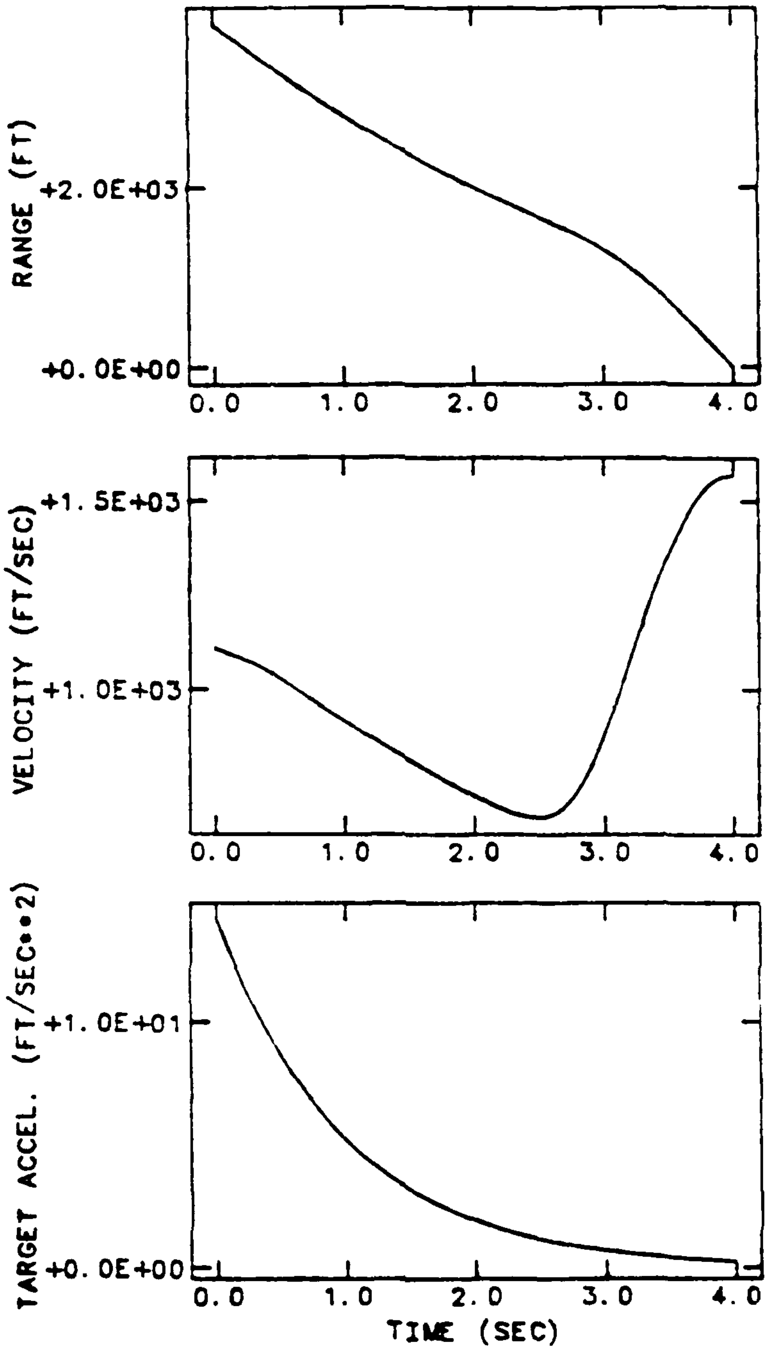


Figure 6.50, Range, Velocity, and Target Acceleration
$\Delta\bar{A} = 0$

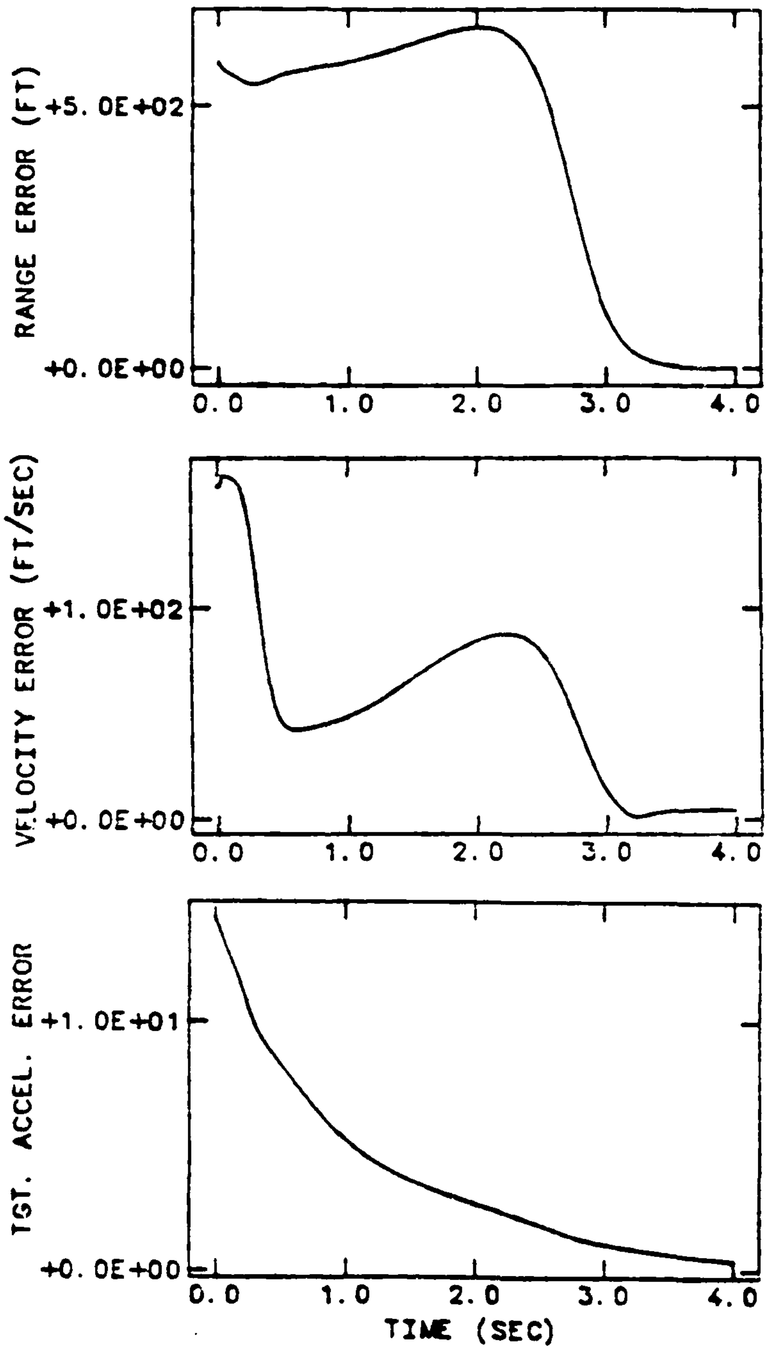


Figure 6.51, Range, Velocity, and Target Acceleration Errors $\Delta\bar{A} = 0$

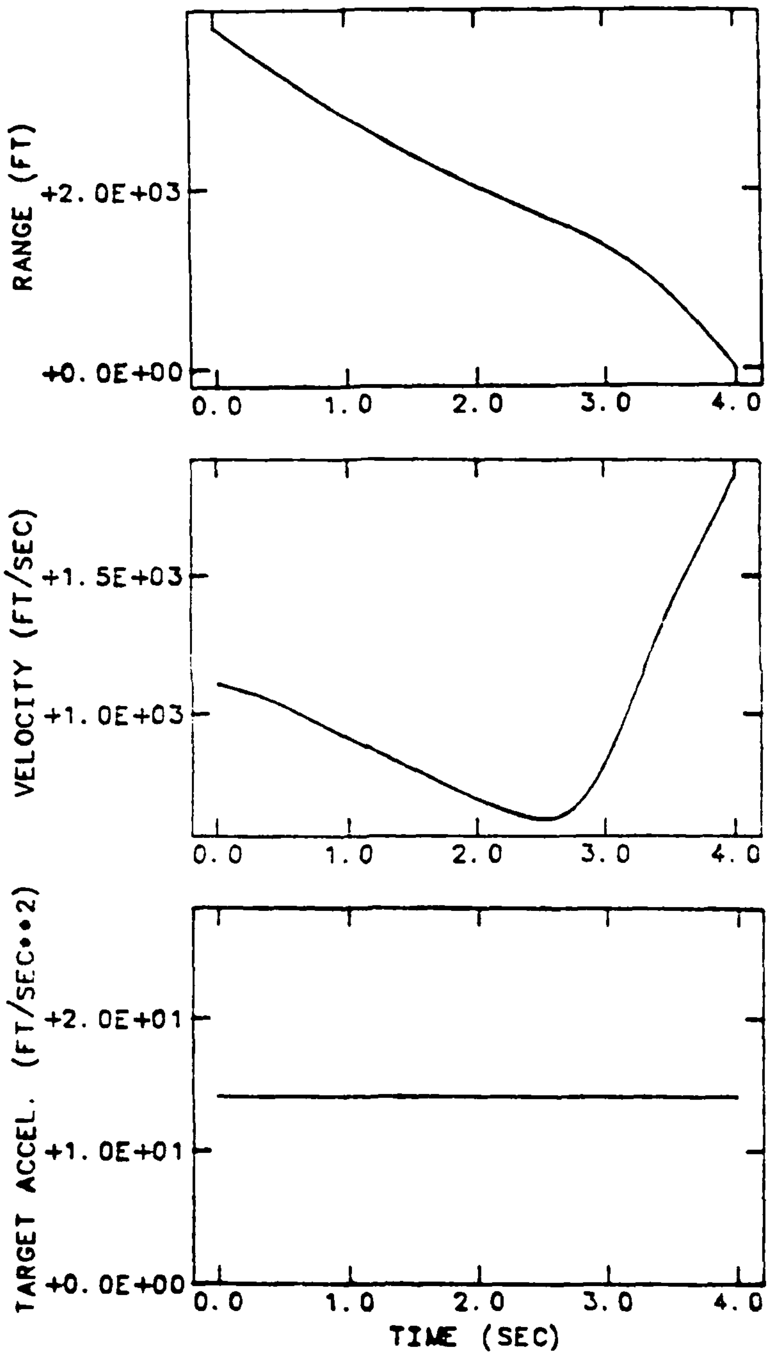


Figure 6.52, Range, Velocity, and Target Acceleration
$\underline{\Delta \bar{A}} = 1$

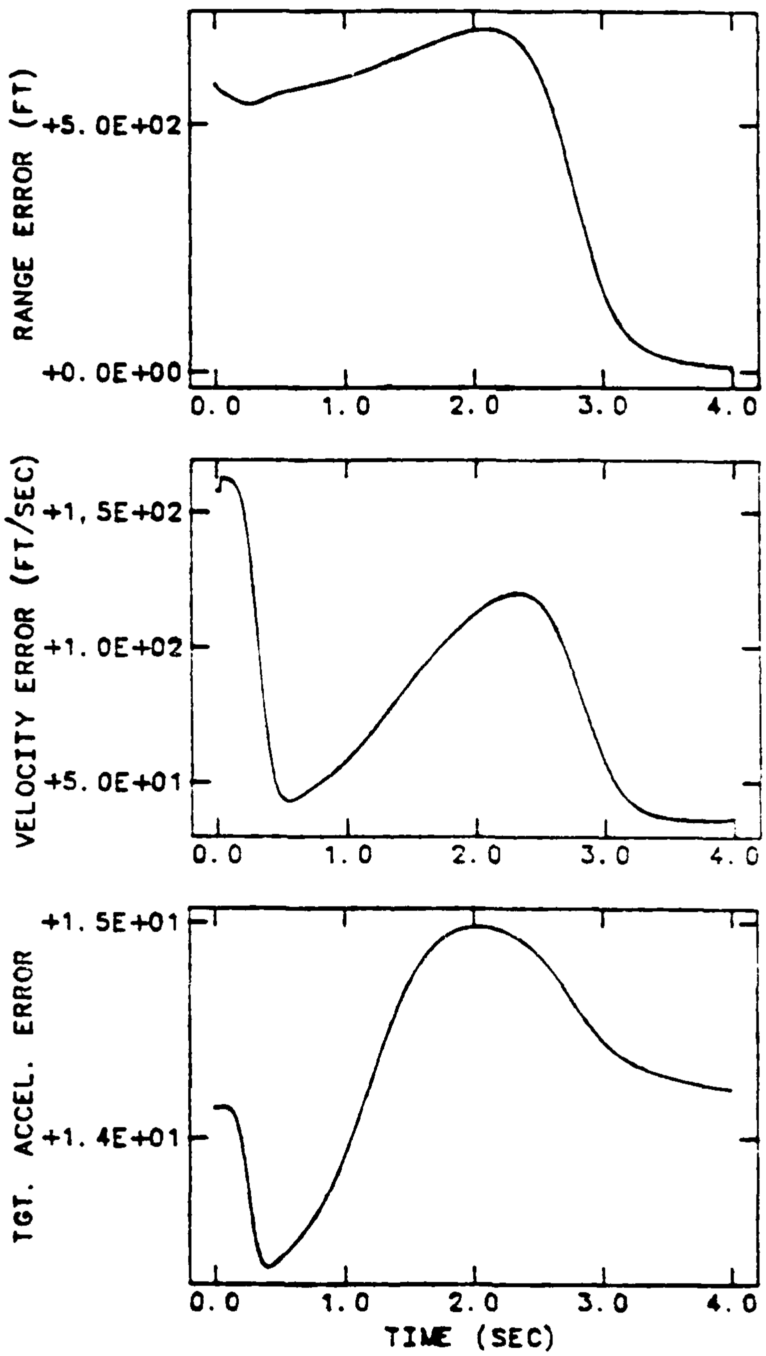


Figure 6.53, Range, Velocity, and Target Acceleration Errors $\Delta\bar{A} = 1$

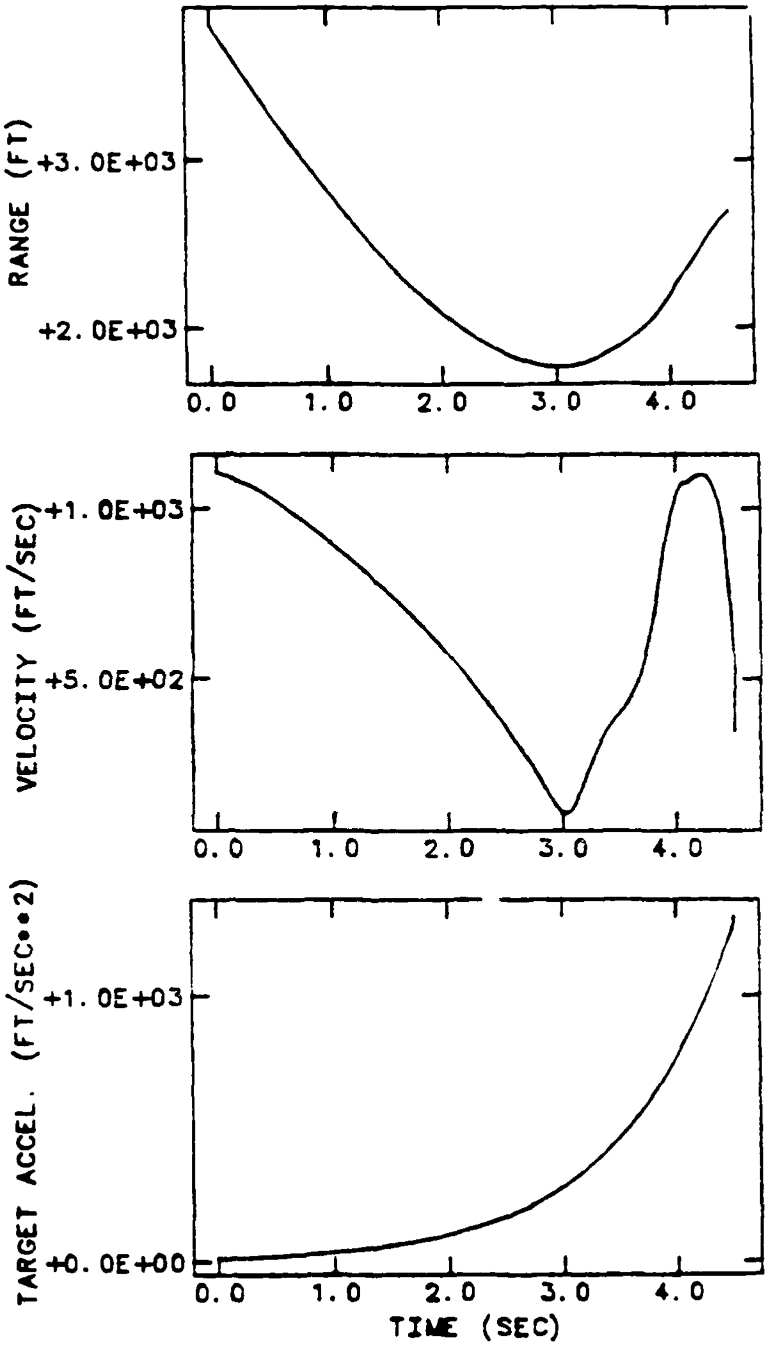


Figure 6.54, Range, Velocity, and Target Acceleration
$\angle\bar{A} = 2$

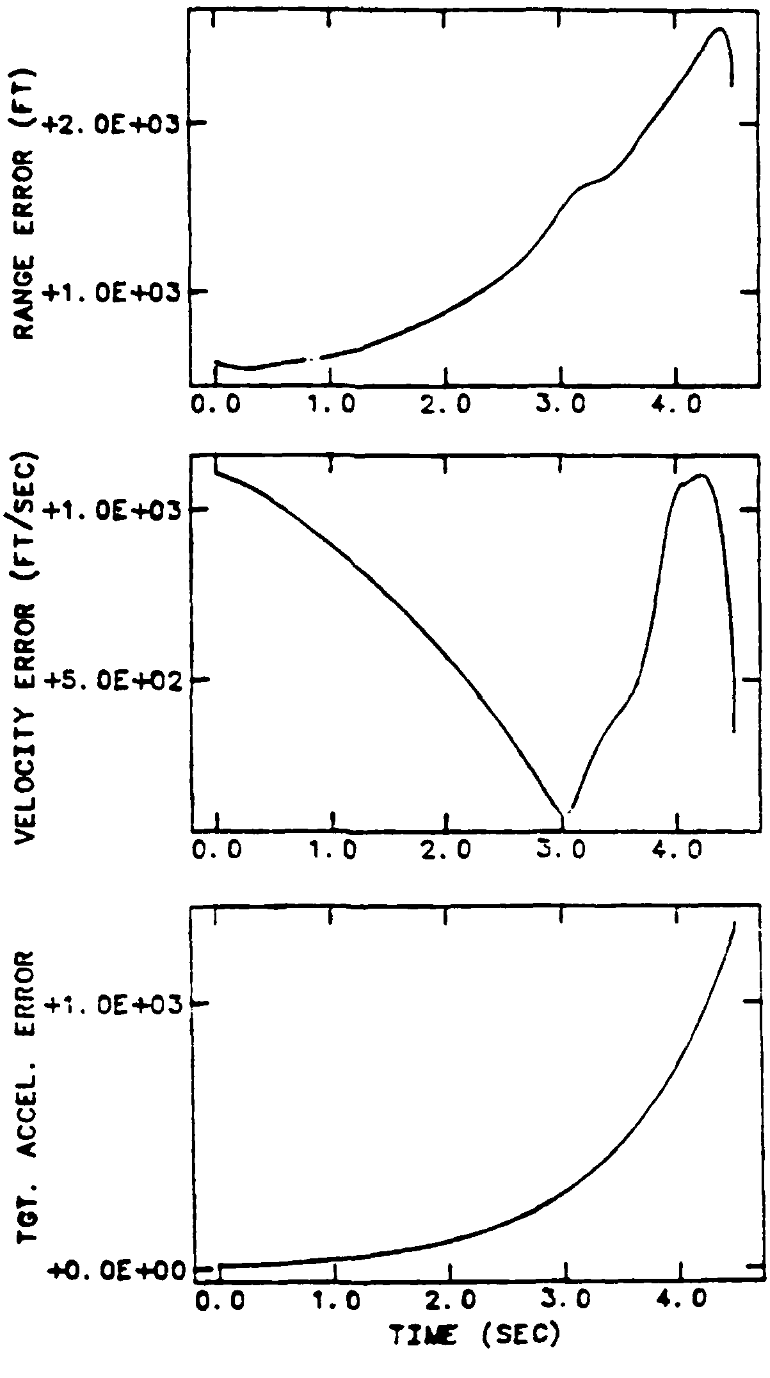


Figure 6.55, Range, Velocity, and Target Acceleration Errors $\Delta\bar{A} = 2$

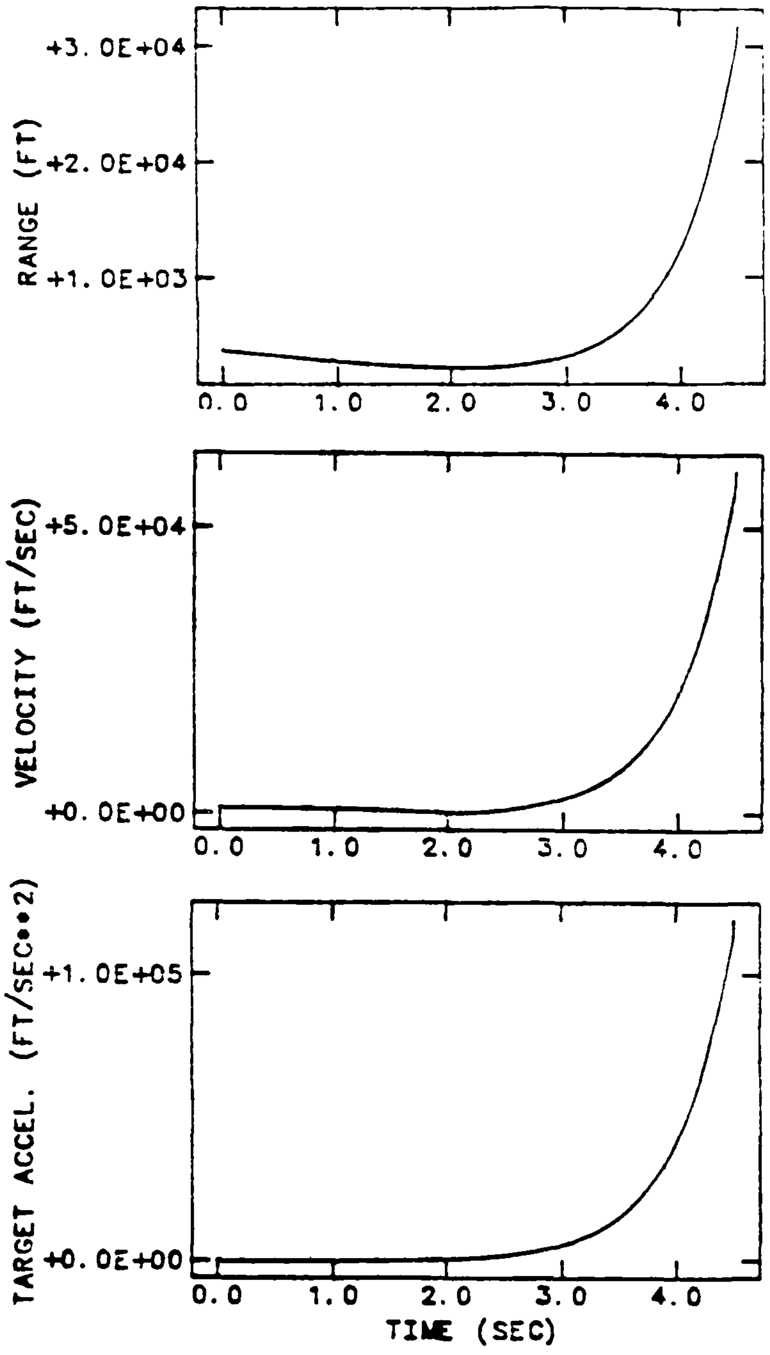


Figure 6.56, Range, Velocity, and Target Acceleration
$\triangle\bar{A} = 3$

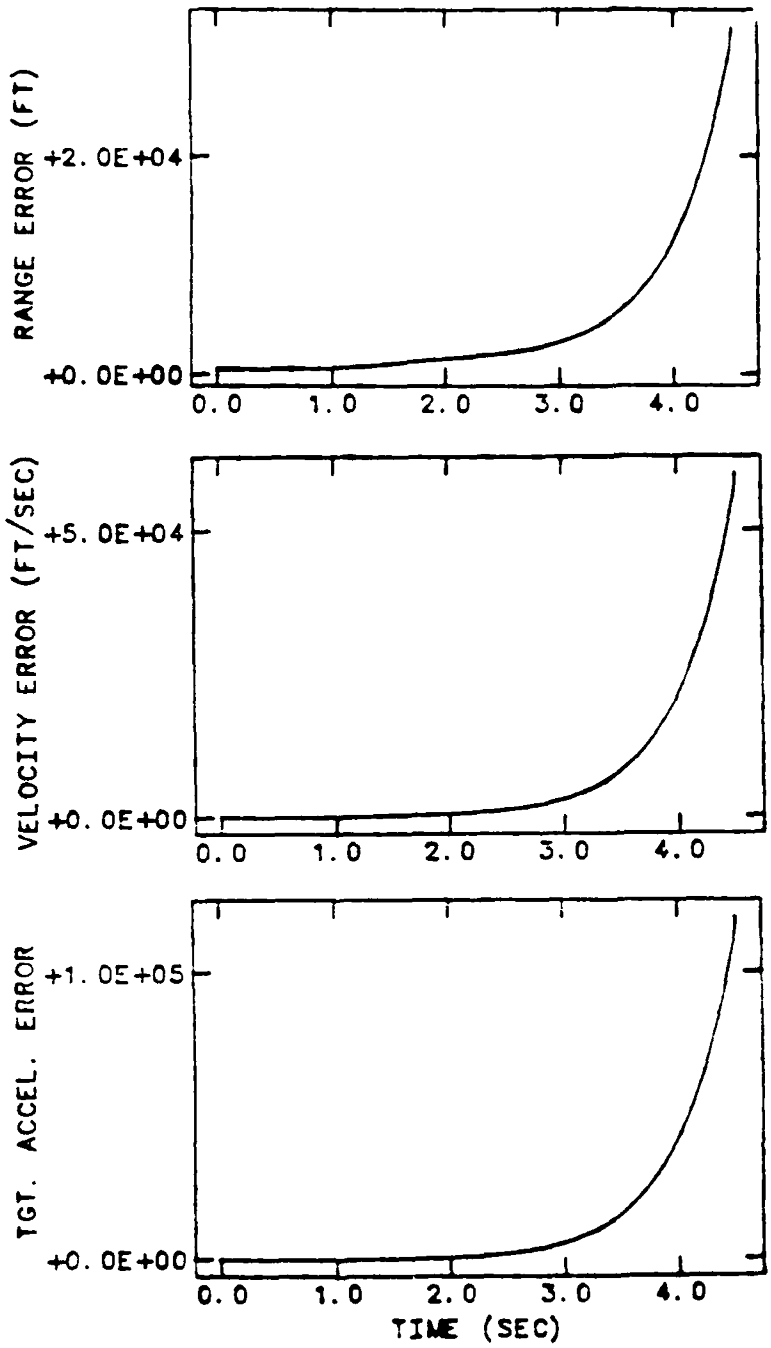


Figure 6.57, Range, Velocity, and Target Acceleration Errors $\underline{\Delta A}$ = 3

6.4.3 Observer Performance Improvements

The purpose of this section is to demonstrate the usefulness of the LQG guidance law, derived to minimize terminal miss distance as well maximize the observability Grammian matrix of the PMO (equation 5.22-5.25). To simplify the analysis, a 2-dimensional system model is used. For a 2-dimensional system, the measurement model comes from equations (6.62) and (6.63). For the PMO, the measurement model is

$$y = H(z)x \tag{6.79}$$

where

$$H(z) = [\ \sin\theta\ ,\ -\cos\theta\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ 0\ ] \tag{6.80}$$

Following the same steps in Section 5.3, Q becomes

$$Q = \begin{bmatrix} 2\sigma^{-1}I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.81}$$

which is a 6x6 matrix.

Given the following intercept geometry

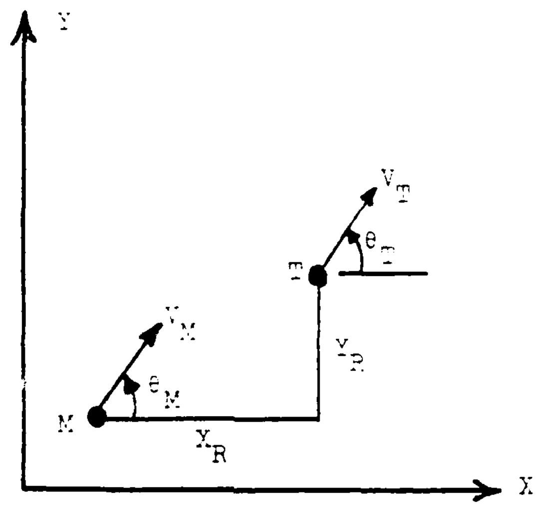


Figure 6.58, Intercept Geometry

the following initial launch conditions are selected to closely match those by Hull, Speyer, Tseng, and Larson [63]: Initial range of 3000 ft., missile velocity of 390 ft./sec., target velocity of 300 ft./sec., target direction of θ_T = 30 deg., and zero target acceleration.

Figures (6.59) and (6.60) show the results of using pro-nav guidance and the LQG guidance law (equation 6.55), respectively. In both cases, the main goal is to hit the target.

To solve the LQG guidance law which increases the PMO's observability Grammian matrix (equations 5.11-5.12), the differential equations (equations 5.22-5.25) have to be solved backward in time from t_f to

t_0. The first step is to use the following approximation for t_f, since it is not readily available.

$$t_f = -\frac{R}{\dot{R}} \tag{6.82}$$

With this t_f, the guidance law is solved and implemented in the simulation, where the results are on Figure (6.61) for $\alpha = .667$. The missile swings past the line-of-sight to the target and then comes back. This is similiar to the results of Hull, Speyer, Tseng, and Larson [63], except that the missile overshoots the target at the end. This is because t_f is an approximation (equation 6.73) and is only solved once.

The next step is to update t_f periodically, as is done in the LQG guidance law, and resolve the new guidance law each time. The new results, shown in Figure (6.62), show a similar trajectory with the exception that the missile hits the target.

Increasing the PMO's observability Grammian means decreasing the PMO's error variance. To show if this new guidance law decreases the PMO's error variance matrix (equation 5.11), a time-plot is generated of the maximum eigenvalue of the error variance matrix of the PMO, with both the standard LQG guidance law and the new LQG guidance law. The results, (Figure (6.63)), show that the error variance is reduced by the new LQG guidance law, as would be expected. The minimum eigenvalue shows the same trend.

In addition, the minimum and maximum eigenvalues of equation (4.65) for the Lyapunov function derived without parameter uncertainties are generated for the new guidance law using the PMO and the LQG guidance law using the same observer. Figure 6.64 shows that the minimum eigenvalue of the Lyapunov equation for the new guidance law has a slightly larger negative slope and more positive value, but still remains bounded. The increased slope is due to the fact that the system initially diverges to improve observability. Figure 6.65 shows the same trend for the maximum eigenvalue.

The results show that the guidance law causes the missile to maneuver in such a way as to improve the observability of the nonlinear measurements with a slight deterioration to the Lyapunov funtion. The end result is a guidance law that still hits the target, and in addition improves the PMO's performance by increasing its observability Grammian matrix. The fact that the maximum eigenvalue of the Lyapunov equation has a larger slope indicates that the convergence is faster than that of the LQG guidance law. The larger positive value in the beginning is due to the missile's initial deviation from the target.

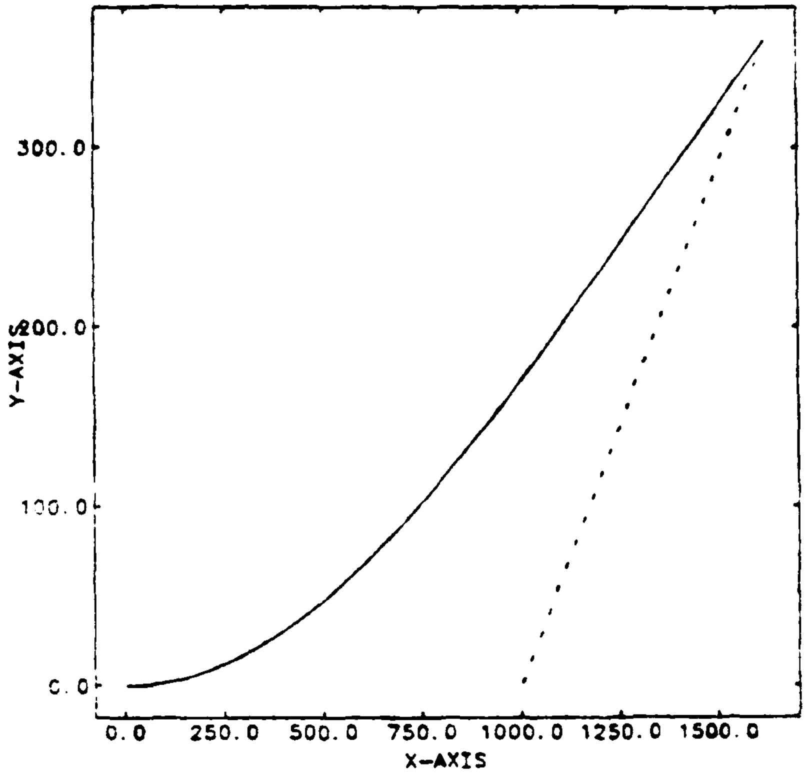


Figure 6.59, , Pro-Nav Guidance

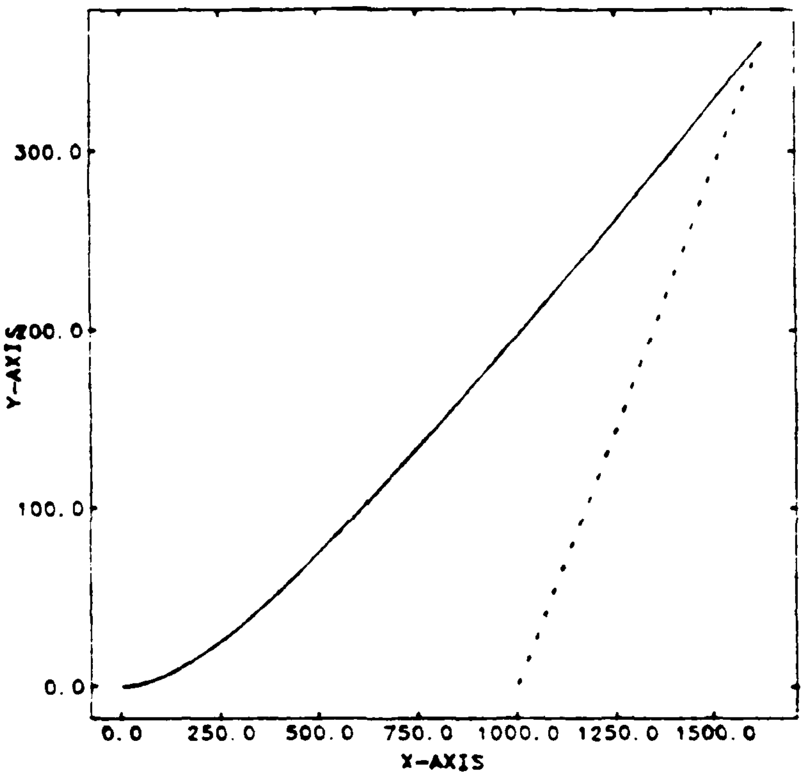


Figure 6.60, LQG Guidance

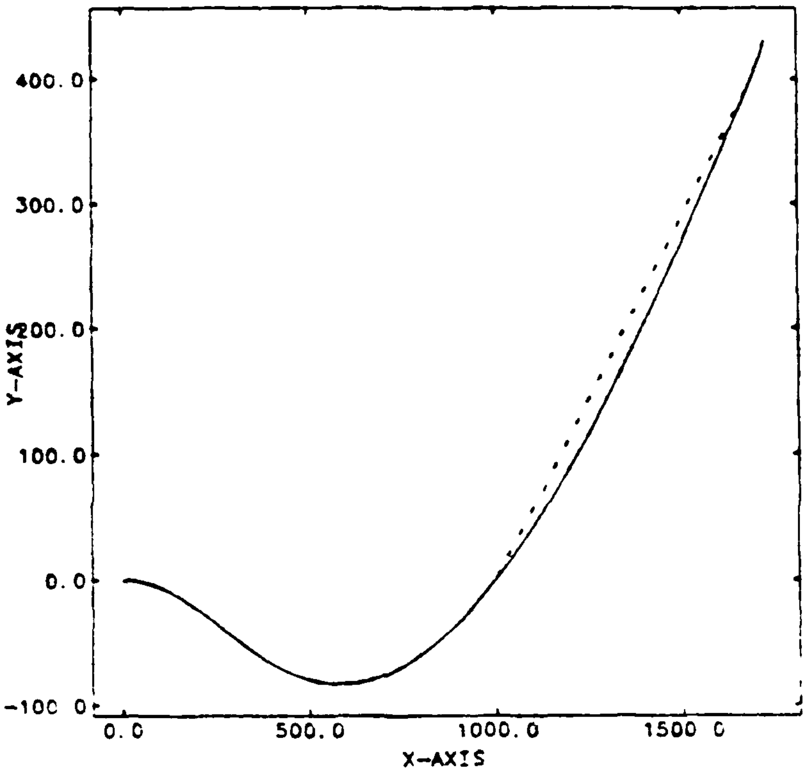


Figure 6.61, LQG With Error Variance Reduction Guidance

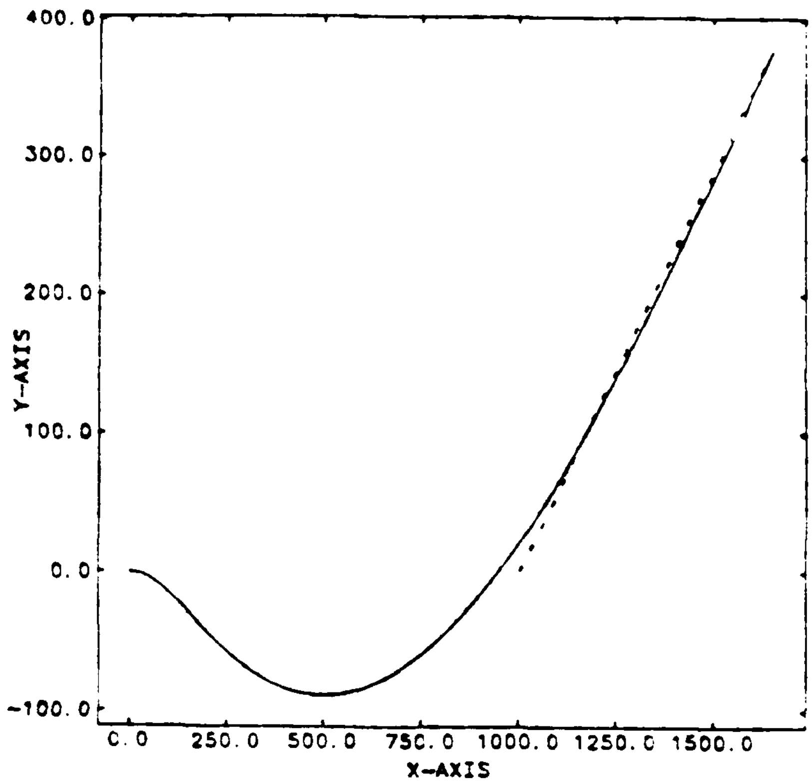


Figure 6.62, LQG With Error Variance Reduction Guidance, t_f Updated

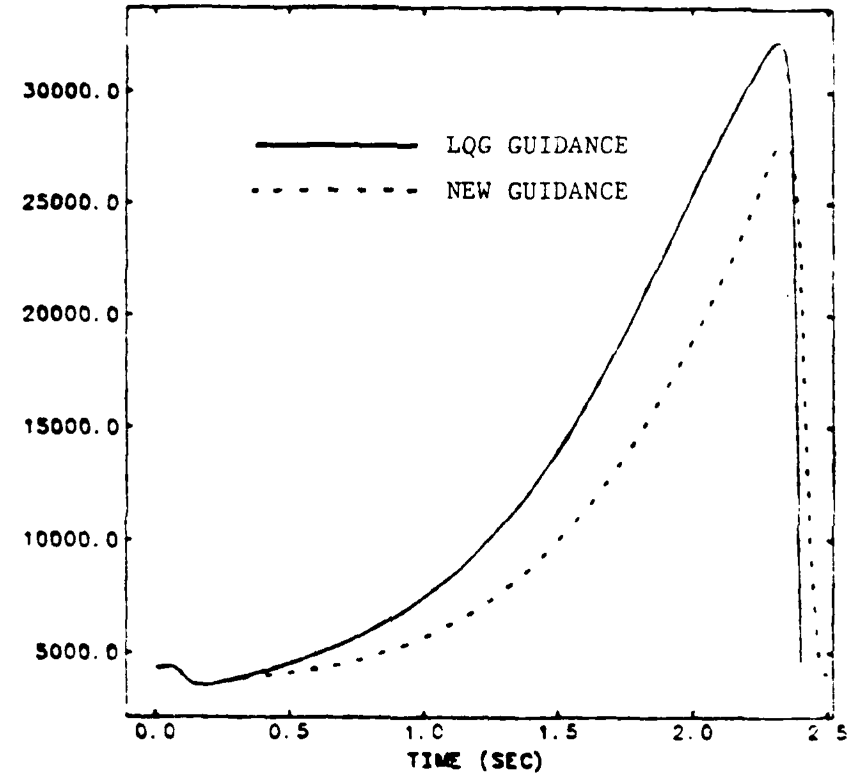


Figure 6.63, Maximum Eigenvalue of the Error Variance Matrix

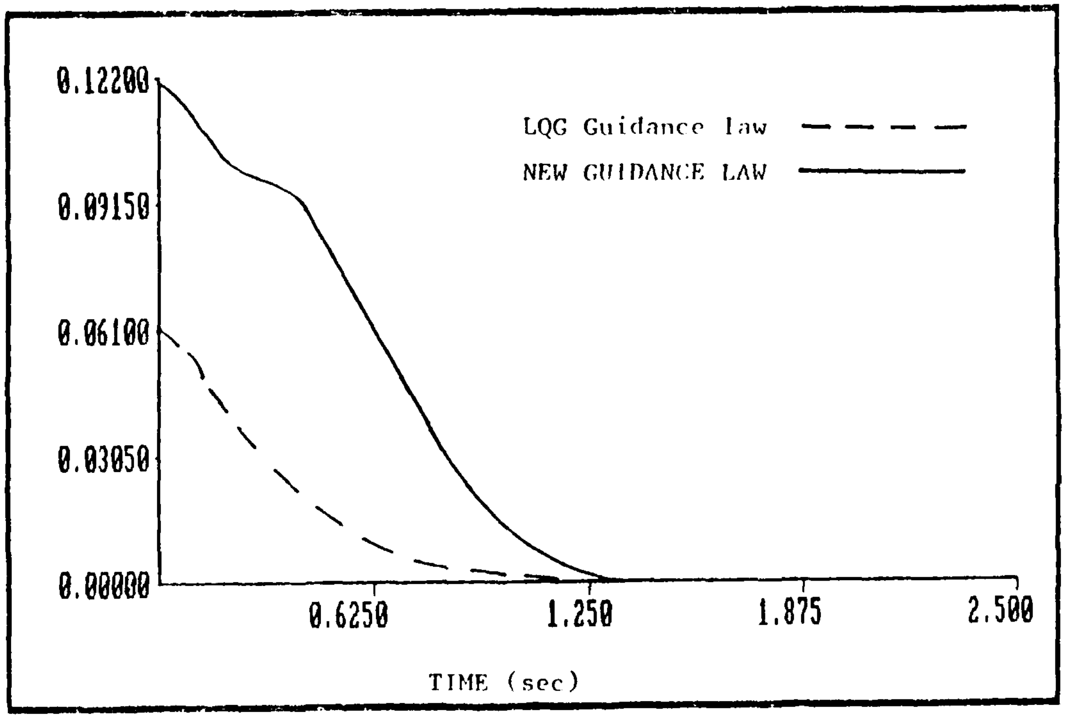


Figure 6.64, Minimum Eigenvalue of the Lyapunov Equation for the Lyapunov Function Derived with Parameter Uncertainties

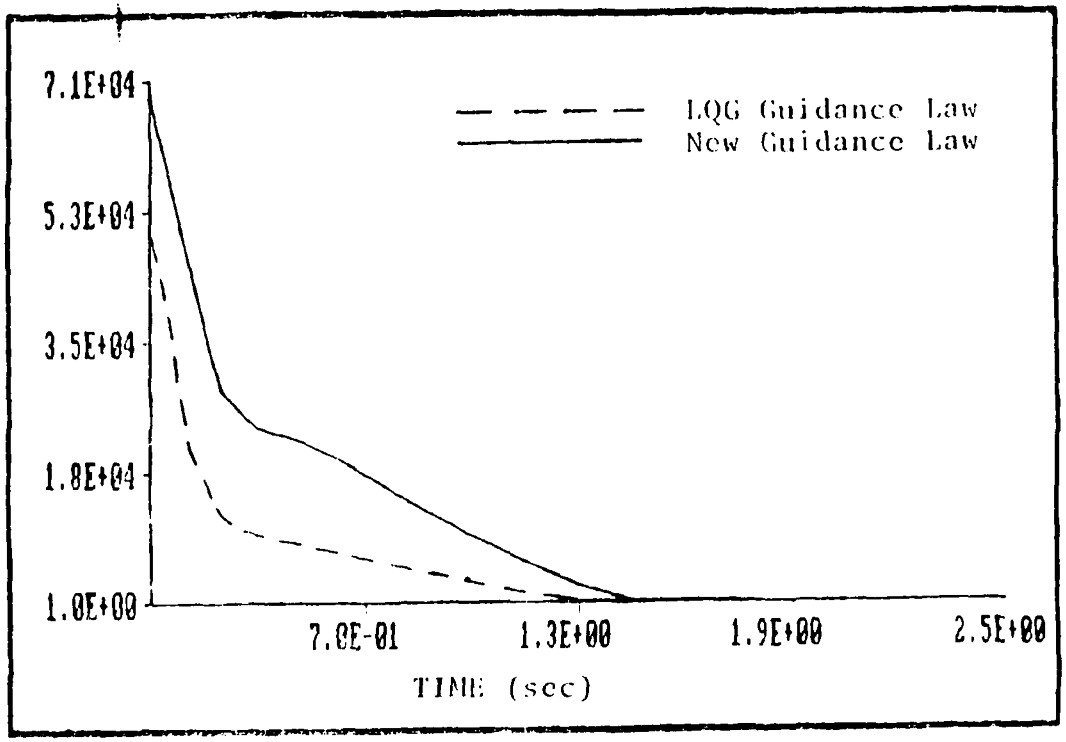


Figure 6.65, Maximum Eigenvalue of the Lyapunov Equation for the Lyapunov Function Derived with Parameter Uncertainties

SECTION VII

CONCLUSIONS

Given the linear, time-varying closed-loop system with an observer in the loop, several Lyapunov functions are derived for the first time, to show that these systems are stable in the sense of Lyapunov. The Lyapunov functions are used to provide a measure of performance, independent of the path taken, for the linear, finite-time problem, and certain classes of nonlinear, finite-time problems like the homing missile problem.

The Lyapunov function which consists of adding the controller Lyapunov function by Anderson and Moore [6] to the observer Lyapunov function by Song and Speyer [119,120] is not valid for all controller/observer systems. However, the controller performance index is scaled such that the combined Lyapunov functions are valid without affecting the control gain. Further, this Lyapunov function is used as a means of improving the stability of the controller/observer system through an overall design selection of the controller and observer parameters to meet the Lyapunov function requirements. This is demonstrated in Section 6.2.2.3; where the controller gain is designed based on the combined Lyapunov function.

Since the combined Lyapunov function is not valid for all controller/observer systems, a Lyapunov function is derived for the cascaded system. The result is a Lyapunov function which consists of the separate controller and observer Lyapunov functions and an additional term, which is a coupling of the system states and the observer errors. This Lyapunov function is valid for all controller/observer systems. When system parameter uncertainties are introduced, this Lyapunov function is not very useful for identifying system stability. This is shown in Sections 6.2.1, 6.2.2, 6.3.3, and 6.4.2.

A Lyapunov function is derived to directly account for system parameter variations. This Lyapunov function is very accurate in identifying system stability of the linear, time-invariant system under parameter variations when compared to eigenvalue analysis. This Lyapunov function is also useful in providing a measure of system performance for the linear, time-varying, finite-time problem and the homing missile guidance problem. The results of this are in Sections 6.2.1, 6.2.2, 6.3.3, and 6.4.2.

The control law which is designed for the missile guidance problem to minimize terminal miss as well as improve the performance of an observer in the loop causes the missile to maneuver in such a way as to increase the observability Grammian matrix of the observer and still hit the target. The results are very close to those by Hull, Speyer, Tseng, and Larson [63]. The Lyapunov function from Section III, which is used as the basis for the derivation of this guidance law, shows an improvement in performance over the linear quadratic Gaussian guidance law. The main contribution is that a closed-loop solution of the control law is obtained.

There are several limitations to the usefulness of the Lyapunov functions derived for the linear, time-varying controller/observer cascaded system, First, they are only valid for the deterministic systems. Second, they can only be used to determine system stability for the infinite-time problem. Even then, it is a sufficiency condition for stability. For the finite-time problem, the Lyapunov functions can only provide a measure of system performance. Third, the performance measure is determined by solving backward Lyapunov equations. This requires a fairly good estimate of the final time for the system of interest. For the homing missile guidance problem, an estimate of final time is relatively easy to obtain. And fourth, Lyapunov functions are not unique. There are several Lyapunov functions derived in this dissertation, some of which are more useful than others. There may be a Lyapunov function that is even better suited for this type of system.

There is a need for future research in stability analysis of closed-loop systems with an observer in the loop. The Lyapunov functions should be expanded to stochastic systems, which have a more practical meaning. Since the Lyapunov function is not unique, the development of a Lyapunov function for other aspects of system performance (other than parameter uncertainties) should be considered.

APPENDIX A

INVERSE OF DISCRETE ERROR COVARIANCE (P_K^{-1})

Given the discrete Kalman filter equations

$$\hat{x}_{K+1} = \bar{x}_K + K_K[y_K - H_K\bar{x}_K] \tag{A.1}$$

$$y_K = H_K x_K + \nu_K \quad , \quad \nu_K \sim N(0, R_K) \tag{A.2}$$

define

$$P_K = E[(x_K - \hat{x}_K)(x_K - \hat{x}_K)^T] \tag{A.3}$$

$$\bar{P}_K = E[(x_K - \bar{x}_K)(x_K - \bar{x}_K)^T] \tag{A.4}$$

Substituting equations (A.1) and (A.2) into (A.3) results in

$$\begin{aligned} P_K = E[&(x_K-\bar{x}_K-K_K(H_K x_K+\nu_K-H_K\bar{x}_K)) \\ &*(x_K-\bar{x}_K-K_K(H_K x_K+\nu_K-H_K\bar{x}_K))^T] \end{aligned} \tag{A.5}$$

$$= E[((I-K_K H_K)(x_K-\bar{x}_K)-K_K\nu_K)((I-K_K H_K)(x_K-\bar{x}_K)-K_K\nu_K)^T] \tag{A.6}$$

$$= E[(I-K_K H_K)(x_K-\overline{x}_K)(x_K-\overline{x}_K)^T(I-K_K H_K)^T] + E[K_K \nu_K \nu_K^T K_K^T]$$

$$- E[K_K \nu_K (x_K-\overline{x}_K)^T (I-K_K H_K)^T]$$

$$- E[(I-K_K H_K)(x_K-\overline{x}_K)\nu_K^T K_K^T] \qquad (A.7)$$

Carrying through the expected value and noting that the measurement noise (ν_K) is uncorrelated with the states, x_K, the result is the Joseph-Bucy form of the update equation for the discrete Kalman filter

$$P_K = (I - K_K H_K)\overline{P}_K(I - K_K H_K)^T + K_K R_K K_K^T \qquad (A.8)$$

where

$$\overline{P}_K = A_{K-1} P_{K-1} A_{K-1}^T + Q_{K-1} \qquad (A.9)$$

Combining equations (A.8) and (A.9) results in

$$P_K = (I - K_K H_K) A_{k-1} P_{K-1} A_{K-1}^T (I - K_K H_K)^T$$

$$+ (I - K_K H_K) Q_K (I - K_K H_K)^T + K_K R_K K_K^T \qquad (A.10)$$

Define

$$A_K \triangleq (I - K_K H_K) A_{K-1} \qquad (A.11)$$

$$\gamma_K \triangleq (I - K_K H_K) Q_K (I - K_K H_K)^T + K_K R_K K_K^T \qquad (A.12)$$

such that equation (A.10) becomes

$$P_K = A_K P_{K-1} A_K^T + \gamma_K \qquad (A.13)$$

The Lyapunov function for the observer is selected as

$$V_K(x_K, e_K, t_K) = e_K^T P_K^{-1} e_K \qquad (A.14)$$

where

$$e_{K+1} = A_K e_K \qquad (A.15)$$

and

$$P_{K-1}^{-1} = A_K^T P_K^{-1} A_K + Q_{e_K} \qquad (A.16)$$

where Q_{e_K} is derived by developing ΔV_K as follows:

$$\Delta V_K = V_{K+1} - V_K$$

$$= e_K^T [A_K^T P_{K+1}^{-1} A_K - P_K^{-1}] e_K \qquad (A.17)$$

Substituting equations (A.8) and (A.9) into equation (A.17) for P_{K+1}^{-1} results in

$$\Delta V_K = e_K^T [A_K^T [A_K P_K A_K^T + \gamma_K]^{-1} A_K - P_K^{-1}] e_K \qquad (A.18)$$

By assuming (A_K, Γ_K) is controllable and (A_K, H_K) is observable, the system described by equation (A.15) is asymptotically stable. Therefore, e_K must converge to zero and A_K is nonsingular. Equation (A.18) becomes

$$\Delta V_K = e_K^T[(P_K + A_K^{-1}\gamma_K A_K^{-T})^{-1} - P_K^{-1}]e_K \qquad (A.19)$$

Applying the matrix inversion lemma to the right side of equation (A.19) results in

$$\Delta V_K = e_K^T[-P_K^{-1}A_K^{-1}(\gamma_K^{-1} + A_K^{-T}P_K^{-1}A_K^{-1})^{-1}A_K^{-T}P_K^{-1}]e_K \qquad (A.20)$$

which is negative definite for $e_K \neq 0$. Therefore, Q_{e_K} becomes

$$Q_{e_K} = P_K^{-1}A_K^{-1}(\gamma_K^{-1} + A_K^{-T}P_K^{-1}A_K^{-1})^{-1}A_K^{-T}P_K^{-1} \qquad (A.21)$$

APPENDIX B

STOCHASTIC DIFFERENTIAL / DIFFERENCE EQUATIONS

Continuous-Time Problems:

Let $\phi(x,t)$ be a scalar real function continuously differentiable in t and having second mixed partial derivatives with espect to x, then the differential $d\phi$ of ϕ is

$$d\phi = \phi_t dt + \phi_x dx + 0.5\,\mathrm{tr}(GQG^T\phi_{xx})dt \tag{B.1}$$

for the stochastic differential equation

$$dx = f(x,t)dt + G(x,t)d\beta \tag{B.2}$$

$$E[d\beta d\beta^T] = Qdt \tag{B.3}$$

The stochastic differential equations for the closed-loop system are

$$\dot{x} = (A-BL)x + BLe + \omega \quad , \quad \omega \sim N(0,\bar{Q}) \tag{B.4}$$

$$\dot{e} = (A-KH)e - K\nu + \omega \quad , \quad \nu \sim N(0,\bar{R}) \tag{B.5}$$

Rewriting these equations in a more general form

$$dx = [(A-BL)x + BLe]dt + d\beta \tag{B.6}$$

$$de = (A-KH)e\,dt - K\,d\nu + d\beta \qquad (B.7)$$

where $d\beta = \omega\,dt$, $d\nu = \nu\,dt$.

β and ν are brownian motion processes with the following properties

$$E[d\beta] = E[d\nu] = 0 \qquad (B.8)$$

$$E[d\beta\, d\beta^T] = \bar{Q}\,dt \qquad (B.9)$$

$$E[d\nu\, d\nu^T] = \bar{R}\,dt \qquad (B.10)$$

$$E[d\beta\, d\nu^T] = 0 \qquad (B.11)$$

given the following definitions

$$X \equiv E[xx^T] \qquad (B.12)$$

$$S \equiv E[xe^T] \qquad (B.13)$$

$$P \equiv E[ee^T] \qquad (B.14)$$

Applying equation (B.1) to (B.12) first results in

$$dX = dE[xx^T] = Ed[xx^T]$$

$$= E\{([(A-BL)x + BLe]dt + d\beta)x^T$$

$$+ x([(A-BL)x + BLe]dt + d\beta)^T$$

$$+ d\beta d\beta^T dt\} \qquad (B.15)$$

Carrying the expectation through, using the definitions (B.12)-(B.14), equation (B.15) becomes

$$dX = [(A-BL)X + X(A-BL)^T + BLS^T + SL^TB^T + \overline{Q}]dt \qquad (B.16)$$

which can be rewritten as

$$\dot{X} = \frac{dX}{dt} = (A-BL)X + X(A-BL)^T + BLS^T + SL^TB^T + \overline{Q} \qquad (B.19)$$

Applying equation (B.1) to (B.13) results in

$$dS = dE[xe^T] = Ed[xe^T] \qquad (B.20)$$

$$= E\{([(A-BL)x + BLe]dt + d\beta)e^T$$

$$+ x((A-KH)edt - Kd\nu + d\beta)^T + \overline{Q}dt\} \qquad (B.21)$$

$$= E\{[(A-BL)xe^T + BLee^T]dt + d\beta e^T$$

$$+ xe^T(A-KH)^T dt - xd\nu^T K^T + xd\beta^T + \overline{Q}dt\} \qquad (B.22)$$

Carrying through the expectations, using the definitions (B.12)-(B.14), equation (B.22) becomes

$$dS = [(A-BL)S + BLP]dt + S(A-KH)^T dt + \bar{Q}dt \quad (B.23)$$

which can be rewritten as

$$\dot{S} = \frac{dS}{dt} = (A-BL)S + S(A-KH)^T + BLP + \bar{Q} \quad (B.24)$$

Applying equations (B.1) to (B.14) results in

$$dP = dE[ee^T] = Ed[ee^T] \quad (B.25)$$

$$= E\{[(A-KH)edt - Kd\nu + d\beta]e^T$$

$$+ e[(A-KH)edt - Kd\nu + d\beta]^T$$

$$+ (K\bar{R}K^T + \bar{Q})dt\} \quad (B.26)$$

$$= E\{(A-KH)ee^T dt - Kd\nu e^T d\beta e^T$$

$$+ ee^T(A-KH)^T dt - ed\nu^T K^T + ed\beta^T$$

$$+ (K\bar{R}K^T + \bar{Q})dt\} \quad (B.27)$$

Carrying the expectation through results in

$$dP = (A-KH)Pdt + P(A-KH)^T dt + (K\bar{R}K^T + \bar{Q})dt \quad (B.28)$$

which can be rewritten as

$$\dot{P} = \frac{dP}{dt} = (A-KH)P + P(A-KH)^T + K\bar{R}K^T + \bar{Q} \quad (B.29)$$

Equations (B.19),(B.24), and (B.29) are the constraint equations for the continuous-time optimization problem.

Discrete-Time Problem:

The stochastic difference equations for the closed-loop system are

$$x_{K+1} = \bar{A}_K x_K + B_K L_K e_K + w_K \quad (B.30)$$

$$e_{K+1} = \tilde{A}_K e_K - K_{K+1} v_{K+1} + w_K \quad (B.31)$$

where

$$\bar{A}_K \triangleq A_K - B_K L_K \quad (B.32)$$

$$\tilde{A}_K \triangleq A_K - K_{K+1} H_{K+1} A_K \quad (B.33)$$

$$w_K \sim N(0,\bar{Q}_K) \quad , \quad v_K \sim N(0,\bar{R}_K) \quad (B.34)$$

Given the following

$$X_K \triangleq E[x_K x_K^T] \quad (B.35)$$

$$X_{K+1} = E[x_{K+1}x_{K+1}^T] \tag{B.36}$$

$$= E\{[\bar{A}_K x_K + B_K L_K e_K + \omega_K]$$

$$*[\bar{A}_K x_K + B_K L_K e_K + \omega_K]^T\} \tag{B.37}$$

$$= E\{\bar{A}_K x_K x_K^T \bar{A}_K^T + \bar{A}_K x_K e_K^T L_K^T B_K^T + \bar{A}_K x_K \omega_K^T]$$

$$+ B_K L_K e_K x_K^T \bar{A}_K^T + B_K L_K e_K e_K^T L_K^T B_K^T$$

$$+ B_K L_K e_K \omega_K^T + \omega_K x_K^T \bar{A}_K^T + \omega_K e_K^T L_K^T B_K^T + \omega_K \omega_K^T\} \tag{B.38}$$

and carrying through the expectation in equation (B.38) results in

$$X_{K+1} = \bar{A}_K X_K \bar{A}_K^T + B_K L_K S_K^T \bar{A}_K^T + \bar{A}_K S_K L_K^T B_K^T$$

$$+ B_K L_K P_K L_K^T B_K^T + \bar{Q}_K \tag{B.39}$$

Given

$$S_K \triangleq E[x_K e_K^T] \tag{B.40}$$

$$S_{K+1} = E[x_{K+1} e_{K+1}^T] \tag{B.41}$$

$$= E\{[\bar{A}_K x_K + B_K L_K e_K + \omega_K][\bar{A}_K e_K - K_{K+1}(V_{K+1} + \omega_K]^T\} \tag{B.42}$$

$$= E\{\bar{A}_K x_K e_K^T \bar{A}_K^T - \bar{A}_K x_K V_{K+1}^T K_{K+1} + \bar{A}_K x_K \omega_K^T$$

$$+ B_K L_K e_K e_K^T \bar{A}_K^T - B_K L_K e_K V_{K+1}^T K_{K+1}^T + B_K L_K e_K \omega_K^T$$

$$+ \omega_K e_K^T \bar{A}_K^T - \omega_K V_{K+1}^T K_{K+1}^T + \omega_K \omega_K^T\} \quad (B.43)$$

and carrying through the expectation in equation (B.43) results in

$$S_{K+1} = \bar{A}_K S_K \bar{A}_K^T + B_K L_K P_K \bar{A}_K^T + \bar{Q}_K \quad (B.44)$$

Given

$$P_K \equiv E[e_K e_K^T] \quad (B.45)$$

$$P_{K+1} = E[e_{K+1} e_{K+1}^T] \quad (B.46)$$

$$= E\{(\bar{A}_K e_K - K_{K+1} V_{K+1} + \omega_K)(\bar{A}_K e_K - K_{K+1} V_{K+1} + \omega_K)^T\} \quad (B.47)$$

$$= E\{\bar{A}_K e_K e_K^T \bar{A}_K^T - \bar{A}_K e_K V_{K+1}^T K_{K+1}^T + \bar{A}_K e_K \omega_K^T$$

$$- K_{K+1} V_{K+1} e_K^T \bar{A}_K^T + K_{K+1} V_{K+1} V_{K+1}^T K_{K+1}^T - K_{K+1} V_{K+1} \omega_K^T$$

$$+ \omega_K e_K^T \bar{A}_K^T - \omega_K V_{K+1}^T K_{K+1}^T + \omega_K \omega_K^T\} \quad (B.48)$$

and carrying through the expectation in equation (B.48) results in

$$P_{K+1} = A_K P_K A_K^T + K_{K+1} \bar{R}_{K+1} K_{K+1}^T + \bar{Q}_{K+1} \qquad (B.49)$$

Equations (B.39), (B.44), and (B.49) are the constraint equations for the discrete-time optimization problem.

APPENDIX C

LYAPUNOV FUNCTION VIA HAMILTON-JACOBI EQUATION

The performance index is as follows:

$$J = x_f^T G_f x_f + e_f^T T_f e_f$$
$$+ \int_0^{t_f} (x^T Q_c x + e^T Q_e e + u^T R_c u)\,dt \qquad (C.1)$$

subject to

$$\dot{x} = Ax + Bu \qquad (C.2)$$

$$\dot{e} = (A - KH)e \qquad (C.3)$$

where

$$u = -L\hat{x} = -L(x-e) = u^* + Le \qquad (C.4)$$

$$L = R_c^{-1} B^T \Lambda_X \qquad (C.5)$$

and u^* is the optimal control.

Define the Lyapunov function as the optimal return function

$$V = x^T \Lambda_X x + e^T \Lambda_P e \triangleq \min_{u^*}\{J\} = J^0 \qquad (C.6)$$

Using these equations and the Hamilton-Jacobi equation

$$-\frac{\delta J}{\delta t} = \min_{u^*}\{H\} \quad (C.7)$$

where H (the Hamiltonian) is represented by the partial differential equation:

$$H = \frac{\delta J^o}{\delta x}\dot{x} + \frac{\delta J^o}{\delta e}\dot{e}$$

$$+ x^T Q_c x + e^T Q_e e + u^T R_c u \quad (C.8)$$

the differential equations for Λ_X and Λ_P are derived by equating like-terms in equation (C.7). First, solving equation (C.8) results in

$$H = 2x^T \Lambda_X (Ax+Bu) + 2e^T \Lambda_P (A-KH)e$$

$$+ x^T Q_c x + e^T Q_e e + u^T Ru \quad (C.9)$$

The minimization with respect to u^* is accomplished by making the following substitution from (C.4)

$$u = -R_c^{-1} B^T \Lambda_X x + Le \quad (C.10)$$

Therefore

$$\min_{u^*}\{H\} = x^T\Lambda_X Ax + x^T A^T\Lambda_X x - 2x^T\Lambda_X BR_c^{-1}B^T\Lambda_X x$$

$$+ 2x^T\Lambda_X BLe + e^T\Lambda_P \bar{A}e + e^T\bar{A}^T\Lambda_P e + x^T Q_c x$$

$$+ e^T Q_e e + (Le-R_c^{-1}B^T\Lambda_X x)^T R_c (Le-R_c^{-1}B^T\Lambda_X x) \qquad (C.11)$$

where $\bar{A} = A - KH$

The left hand side of equation (C.7) is

$$\frac{\partial J}{\partial t} = x^T\dot{\Lambda}_X x + e^T\dot{\Lambda}_P e \qquad (C.12)$$

Equation (C.11) and (C.12) are substituted in Equation (C.7).

$$0 = x^T\dot{\Lambda}_X x + e^T\dot{\Lambda}_P e + e^T\Lambda_P e + x^T\Lambda_X Ax + x^T A^T\Lambda_X x$$

$$- x^T L^T R_c Lx + e^T\Lambda_P \bar{A}e + e^T\bar{A}^T\Lambda_P e + x^T Q_c x$$

$$+ e^T Q_e e + e^T L^T R_c Le \qquad (C.13)$$

or

$$0 = x^T\{\dot{\Lambda}_X + \Lambda_X A + A^T\Lambda_X - L^T R_c L + Q_c\}x$$

$$+ e^T\{\dot{\Lambda}_P + \Lambda_P \bar{A} + \bar{A}^T\Lambda_P + L^T R_c L + Q_e\}e \qquad (C.14)$$

This leads to the following differential equations.

$$\dot{\Lambda}_X = -\Lambda_X A - A^T\Lambda_X + L^T R_c L - Q_c \qquad (C.15)$$

$$\dot{\Lambda}_P = -\Lambda_P \tilde{A} - \tilde{A}^T\Lambda_P - L^T R_c L - Q_e \qquad (C.16)$$

Using equation (C.5), the differential equation for Λ_X can be rewritten as

$$\dot{\Lambda}_X = -\Lambda_X \bar{A} - \bar{A}^T\Lambda_X - L^T R_c L - Q_c \qquad (C.17)$$

where $\bar{A} = A - BL$

Thus, equations (C.16) and (C.17) are the same as those derived in Section III.

With parameter uncertainty in the dynamic equations, equations (C.2) and (C.3) are written as

$$\dot{x} = A'x + B'u \qquad (C.18)$$

$$\dot{e} = Dx + \tilde{A}'e \qquad (C.19)$$

where

$$A' = A_c + (A - A_c) \qquad (C.20)$$

$$B' = B_c + (B - B_c) \qquad (C.21)$$

$$\bar{A}' = A_c - K_cH_c + (B-B_c)L_c - K_c(M-M_c)L_c \qquad (C.22)$$

$$D = (A-A_c) - K_c(H-H_c) - (B-B_c)L_c + K_c(M-M_c)L_c \quad (C.23)$$

Using equations (C.18) and (C.19), the Hamilton-Jacobi equation (C.7) becomes

$$
\begin{aligned}
\emptyset = {} & x^T\dot{\Lambda}_X x + x^T\dot{\Lambda}_S e + E^T\dot{\Lambda}_S^T x + E^T\Lambda_P e \\
& + x^T\Lambda_X A'x + x^TA'^T\Lambda_X x - x^T\Lambda_X B'L_c x - x^TL_c^TB'^T\Lambda_X x \\
& + x^T\Lambda_X B'L_c e + e^TL_c^TB'^T\Lambda_X x + e^T\Lambda_S^T A'x + x^T\bar{A}'^T\Lambda_S e \\
& - e^T\Lambda_S^T B'L_c x - x^TL_c^TB'^T\Lambda_S e + e^T\Lambda_S^T B'L_c e + e^TL_c^TB'^T\Lambda_S^T e \\
& + x^T\Lambda_S Dx + x^TD^T\Lambda_S^T x + e^T\Lambda_P Dx + x^TD^T\Lambda_P e + x^T\Lambda_S\bar{A}'e \\
& + e^T\bar{A}'^T\Lambda_S^T x + e^T\Lambda_P\bar{A}'e + e^T\bar{A}'^T\Lambda_P e + x^TQ_c x + e^TQ_e e \\
& + x^TL_c^TR_cL_c x - x^TL_c^TR_cL_c e - e^TL_c^TR_cL_c x + e^TL_c^TR_cL_c e \quad (C.24)
\end{aligned}
$$

Collecting $x^T\{\ \}x$, $x^T\{\ \}e$, and $e^T\{\ \}e$ terms while noting

$$L_c \neq R_c^{-1}B'^T\Lambda_X \qquad (C.25)$$

results in

$$\dot{\Lambda}_X = -\Lambda_X A' - A'^T\Lambda_X - L_c^T R_c L_c - Q_c$$

$$- \Lambda_S D - D^T\Lambda_S^T - \Lambda_X B' L_c - L_c^T \overline{B}'^T \Lambda_X \qquad (C.26)$$

$$\dot{\Lambda}_S = -\Lambda_S \bar{A}' - A'^T\Lambda_S - L_c^T B'^T \Lambda_S$$

$$- D^T\Lambda_P - \Lambda_X B' L_c + L_c^T R_c L_c \qquad (C.27)$$

$$\dot{\Lambda}_P = -\Lambda_P \bar{A}' - \bar{A}'^T\Lambda_P - \Lambda_S^T B' L_c$$

$$- L_c^T B'^T \Lambda_S^T - Q_e - L_c^T R_c L_c \qquad (C.28)$$

By letting

$$\overline{B} = B' L_c \qquad (C.29)$$

$$\overline{A}' = A' - B' L_c \qquad (C.30)$$

equations (C.26) through (C.28) can be rewritten as

$$\dot{\Lambda}_X = -\Lambda_X \overline{A}' - \overline{A}'^T\Lambda_X - L_c^T R_c L_c$$

$$- Q_c - \Lambda_S D - D^T\Lambda_S^T \qquad (C.31)$$

$$\dot{\Lambda}_S = -\Lambda_S \bar{A}' - \overline{A}'^T\Lambda_S + L_c^T R_c L_c$$

$$- D^T\Lambda_P - \Lambda_X \overline{B} \qquad (C.32)$$

$$\dot{\Lambda}_P = -\Lambda_P A' - A'^T \Lambda_P - L_c^T R_c L_c$$

$$- Q_e - \Lambda_S^T \overline{B} - \overline{B}^T \Lambda_S \qquad (C.33)$$

These equations are the same as those derived in Section IV.

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